THE G ME OF GR PH NIM ON GR PHS WITH FOUR EDGES

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BSTR CT. This work is concerned with the study of the *Game of Graph Nim* – a class of two-player combinatorial games – on graphs with 4 edges. To each edge of such a graph is assigned a positive-integer-valued edge-weight, and during each round of the game, the player whose turn it is to make a move selects a vertex, and removes a non-negative integer edge-weight from each of the edges incident on that vertex, such that (i) the remaining edge-weight on each of these edges is a non-negative integer, (ii) and the total edge-weight removed during a round is strictly positive. The game continues for as long as the sum of the edge-weights remaining on all edges of the graph is strictly positive, and the player who plays the last round wins. n initial configuration of edge-weights is considered *winning* if the player who plays the first round wins the game, whereas it is defined as *losing* if the player who plays the second round wins. In this paper, we characterize, *almost entirely*, all winning and losing configurations for this game on all graphs with precisely 4 edges each. Only one such graph defies our attempt to *fully* characterize the winning and losing configurations of edge-weights on its edges – we are still able to provide a set of partial results pertaining to this graph.

1. Introduction nd Form l Description of the G me of Gr ph Nim

The *Game of Nim* is one of the earliest examples of *two-player combinatorial games*, purported to have originated in ancient China and mentioned in European references that date back to as early as the 16th century. While several variants of this game are in existence, we describe here one of the most commonly studied versions, as follows:

Definition 1.1 (The Game of Nim)**.** The game begins with k piles or heaps of chips, with the i-th pile containing x_i chips for $i \in \{1, 2, \quad, k\}$, where $x_1, x_2, \quad, x_k$ are positive integers. Two players take turns to make moves, where a *move* involves choosing a pile (that is not already empty) and subsequently removing at least one chip from that pile. The game continues until all the piles have been rendered empty, and the player to remove the very last chip is declared the winner.

The set $\{x_1, x_2, \quad, x_k\}$ is referred to as the *initial configuration* or *initial position* of the game, and it is referred to as a *winning position* if the player who plays the first round is destined to win the game, whereas it is referred to as a *losing position* if the player who plays the first round is destined to lose the game. This is a *perfect information game*, as the piles of chips are revealed in their entirety to the two players *before* the game begins.

This game has been studied extensively, and its winning and losing positions have been characterized fully (see, for instance, the seminal work done in Bouton (1901), as well as Conway (2000) and Ferguson (2020) for a more detailed literature on these games). In this paper, in some of our proofs, we have made

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use of the necessary and sufficient condition for an initial configuration to be winning for the Game of Nim defined in Definition 1.1 (we refer the reader to Theorem 2.1 for a description of the same).

The game we study in this paper is inspired by the Game of Nim, and played on a graph. As far as we are aware, this game, henceforth referred to as the *Game of Graph Nim*, was introduced and studied in Williams (2017), and our work in this paper serves as both a significant extension to the results obtained in Williams (2017) and to show that the analysis required herein is far more complicated and *ad hoc* (in the sense that it depends heavily on the underlying graph on which the game is being played) compared to the Game of Nim in Definition 1.1. In what follows, we denote by $G = (V,E)$ an undirected graph whose set of vertices is given by V and whose set of edges is given by E. For $u, v \in V$, we let $\{u,v\}$ denote the edge between these two vertices if they are adjacent (i.e. if $\{u,v\} \in E$). Henceforth, we let $\mathbb{N}$ denote the set of all positive integers and $\mathbb{N}_0$ the set of all non-negative integers.

Definition 1.2 (The Game of Graph Nim)**.** Consider a graph $G = (V,E)$, with an edge-weight $w_0(e)$, where $w_0(e) \in \mathbb{N}_0$, assigned to each $e \in E$. Two players take turns (each such turn is, henceforth, referred to as a *round*) to make moves, where a *move* involves

(i) selecting a vertex $u \in V$,
(ii) removing a non-negative integer weight from each edge $\{u,v\} \in E$ (i.e. each edge in E that is incident on the vertex u), so that the total weight removed is a strictly positive integer, and the remaining edge-weight of each $\{u,v\} \in E$ is a non-negative integer.

The game continues for as long as the edge-weight of each edge in E has not been reduced to 0. The player to play the last round is declared the winner.

Note that this game can be thought of as being played on the complete graph $K_{|V|}$ with $|V|$ many vertices, since for each pair of vertices $u, v \in V$ with no edge present between u and v (i.e. $\{u,v\} \notin E$), it is as if the initial edge-weight $w_0(\{u,v\})$ equals 0. Henceforth, we let the players be referred to as P_1 and P_2, with P_1 assumed to play the first round.

Given the underlying graph $G = (V,E)$, we call $(w_0(e) : e \in E)$ a *winning initial weight configuration* (or, simply, a *winning configuration*) if the player who plays the first round of the game on G, starting with these initial edge-weights, is destined to win. We call $(w_0(e) : e \in E)$ a *losing initial weight configuration* (or, simply, a *losing configuration*) if the player who plays the first round is destined to lose the game.

In this paper, our objective is to consider all graphs comprising precisely 4 edges, i.e. $|E| = 4$, and characterize the winning and losing configurations on each of them. It suffices to consider all *unlabeled* distinct graphs comprising 4 edges each, and barring one, we are able to provide a *complete* characterization of the winning and losing configurations on each of them. For the graph for which we are not able to obtain a full understanding of the winning and losing configurations, we provide partial results that, we believe, are adequate in demonstrating to the reader the challenges faced when analysing the Game of Graph Nim on certain underlying graphs.

1.1. **Organization of the rest of the paper.** Our paper is organized as follows. In §2, we explain how the Game of Graph Nim can be seen as a generalization (that can be played on graphs) of the Game of Nim. This is followed by §3, where we include a brief survey of the existing literature pertinent to the topic this paper is concerned with. In §4, we describe, in detail, the graphs on which we study the Game of Graph Nim in this paper, and the main results we establish pertaining to these underlying graphs. We draw the reader's attention to §4.1 and §4.2, as these two subsections are dedicated to the statements of our main results concerning the two most challenging graphs we study in this paper, namely G_4 and H_1 of Figure 2. The proofs of our main results concerning the graphs F_2, G_2 and G_3 shown in Figure 2 have been provided in §5, the proofs of our main results concerning the graph G_4 have been provided in §6, and finally, in §7, we outline the detailed proofs of *most* of our findings regarding the graph H_1. This is followed by the Appendix,

§8, various parts of which are dedicated to laying down the details omitted from the proofs provided in §5, §6 and §7. For instance, part of the proof of Theorem 4.3 can be found in §8.1, some steps of the proof of Theorem 4.6 have been elaborated upon in §8.2 and §8.3, and several of the claims made in Theorem 4.10, concerning losing initial weight configurations on H_1, have been proved in §8.5, §8.8, §8.9, §8.10, §8.11 and §8.12.

2. THE GAME OF GRAPH NIM GENERALIZES THE GAME OF NIM

We begin §2 by introducing a notation that will be used throughout this paper: henceforth, given any $x \in \mathbb{N}$, we write $x = (a_r a_{r-1} \dots a_1 a_0)_2$ to mean a base-2 representation of x, i.e. $a_r, a_{r-1}, \dots, a_1, a_0$ are values in $\{0,1\}$ such that $x = \sum_{i=0}^{r} a_i 2^i$. Notice that this includes the possibility that there exists some $r' < r$ such that $a_{r'} = 1$ and $a_i = 0$ for each $i \in \{r'+1, \dots, r\}$ – this simply means that we have placed additional 0s to the left of the 'minimal' base-2 representation of x (i.e. the base-2 representation of x that consists of the minimum number of digits), and this is done in order to render x comparable with the base-2 representations of other positive integers.

We now state a well-known result (see, for instance, Brualdi (1977)) that gives a complete description of the winning and losing configurations for the Game of Nim defined in Definition 1.1:

Theorem 2.1. *Consider a Game of Nim that begins with k piles of chips, with the i-th pile containing p_i chips, where $p_1, p_2, \dots, p_k$ are positive integers. Let $p_i = (b_{i,s} b_{i,s-1} \dots b_{i,1} b_{i,0})_2$ for each $i \in \{1,2,\dots,k\}$, where s is chosen such that $b_{i,s} = 1$ for at least one $i \in \{1,2,\dots,k\}$. This configuration is losing if and only if it is* balanced*, i.e. the sum $\sum_{i=1}^{k} b_{i,t}$ is even for each $t \in \{0,1,\dots,s\}$.*

Henceforth, whenever we talk about a k-tuple $(p_1, p_2, \dots, p_k)$, for any $k \in \mathbb{N}$ and any $p_1, p_2, \dots, p_k \in \mathbb{N}_0$, being 'balanced' or 'unbalanced', we interpret $(p_1, p_2, \dots, p_k)$ as the initial configuration for a Game of Nim involving k piles of chips, with the i-th pile containing p_i chips for each $i \in \{1,2,\dots,k\}$.

Theorem 2.1 has consequences when it comes to the Game of Graph Nim, as defined in Definition 1.2, played on a certain class of graphs that we henceforth refer to as *galaxy graphs*. Before we are able to define the notion of galaxy graphs, we must introduce the notion of *star graphs*, which is what we begin Definition 2.2 with. Furthermore, given a finite collection of finite graphs, $\{G_1, G_2, \dots, G_n\}$, with V_i denoting the set of vertices in G_i for each $i \in \{1,2,\dots,n\}$, we call this collection *pairwise-vertex-disjoint* if $V_i \cap V_j = \emptyset$ for all $i, j \in \{1,2,\dots,n\}$ with $i \neq j$.

Definition 2.2. A finite graph $G = (V,E)$ is called a *star graph* if all its edges are incident on a common vertex v_0, i.e. there exists a $v_0 \in V$ such that $\{v, v_0\} \in E$ for each $v \in V \setminus \{v_0\}$, and $\{u,v\} \notin E$ for each pair of distinct vertices $u, v \in V \setminus \{v_0\}$. We refer to v_0 as the *centre* of the star graph (the centre is non-unique if and only if the star graph is just a single edge), and the edge $\{v, v_0\}$ as a *ray* of the star graph for each $v \in V \setminus \{v_0\}$.

We call a graph $G = (V,E)$ a *galaxy graph* if it consists of a finite collection of pairwise-vertex-disjoint star graphs, i.e. V can be partitioned into subsets $V_1, V_2, \dots, V_k$, such that the induced subgraph of G on V_i is a star graph, and there exists no edge between V_i and V_j for any pair of distinct $i, j \in \{1,2,\dots,k\}$. In other words, all connected components of a galaxy graph are star graphs.

Note that, in particular, a graph consisting of a single edge is a star graph, and a graph consisting of a finite collection of pairwise-vertex-disjoint edges is a galaxy graph. Some examples of galaxy graphs have been illustrated in Figure 1. We now state a result, pertaining to the Game of Graph Nim played on galaxy graphs, that follows from Theorem 2.1:

Theorem 2.3. *Consider a galaxy graph G that comprises k pairwise-vertex-disjoint star graphs, the centres of which are $u_1, u_2, \dots, u_k$. Let $u_{i,1}, \dots, u_{i,\ell_i}$ denote the vertices that are adjacent to u_i, for each $i \in \{1,2,\dots,k\}$. The initial weight configuration, $\mathbf{w}_0 = (w_0(\{u_i, u_{i,t}\}) : t \in \{1,2,\dots,\ell_i\}, i \in \{1,2,\dots,k\})$, is*

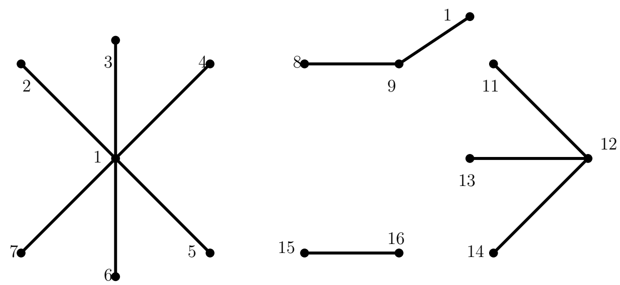

FIGURE 1. n example of a galaxy graph

losing for the Game of Graph Nim played on G if and only if the k-tuple $(p_1, p_2, \ldots, p_k)$, *where* $p_i = \sum_{t=1}^{\ell_i} w_0(\{u_i, u_{i,t}\})$ *for each* $i \in \{1, 2, \ldots, k\}$, *is balanced (as defined in Theorem 2.1).*

Proof. Let us interpret each component of G, i.e. each constituent star-graph in G, as a heap, and the sum p_i of edge-weights assigned to all edges of the i-th star graph as the total number of chips the i-th heap contains at the beginning of the game. player selecting a vertex inside the i-th star graph and removing non-negative-integer-valued edge-weights from each of the edges incident on that vertex, such that the total edge-weight removed, say w, is a strictly positive integer, in the Game of Graph Nim played on G, is equivalent to the player selecting the i-th heap and removing a total of w chips from it in the corresponding Game of Nim. The proof now follows immediately from Theorem 2.1.

Theorem 2.3 also makes it evident that a k-pile Game of Nim can be viewed as a Game of Graph Nim played on a galaxy graph consisting of k pairwise-vertex-disjoint star graphs (or, in other words, consisting of k connected components).

3. LITER TURE REVIEW

The game of Nim has a history spanning centuries, with its origins possibly tracing back to China. It was also documented in European countries during the 16th century. Many variations and generalizations of the classic game of Nim have been studied since the foundational work Bouton (1901), which introduced the nim-sum and provided a complete mathematical solution. Moore (1910) extended this game to Nim_k, where a player may draw from up to k piles out of n piles of objects. Gurvich et al. Gurvich and Ho (2015) analyzes a class of nim games where players remove one token each from up to (or exactly) k non-empty piles. Sprague (1935) and Grundy (1939) independently showed that every impartial game under normal play is equivalent to a Nim heap, leading to the Sprague–Grundy theorem.

The works most closely related to ours are Calkin et al. (2010); Low and Chan (2016); Brown and Williams (2019), which study graph Nim games under the restriction that all edge weights are exactly one. These correspond to the unweighted variant of the graph Nim games considered in our paper. Specifically, Calkin et al. (2010) analyzes such games on paths and caterpillars, Low and Chan (2016) extends the analysis to a broader class of graphs, and Brown and Williams (2019) provides a complete characterization of winning strategies and losing positions for all graphs with four vertices.

Fukuyama (2003 ,B); Erickson (2011) study a variant called Nim on graphs, where a token is moved along adjacent edges and positive integer weights are removed from those edges. Like the game of Nim, the player making the last move wins. Fukuyama (2003) analyzes strategies for various graphs, including bipartite graphs and multigraphs, while Erickson (2011) considers both unit and arbitrary weights on graphs such as complete graphs, the Petersen graph, hypercubes, and even cycles.

Other variants include Bounded Nim (Schwartz (1971)), where moves are limited by an upper bound; CHOMP (Gale (1974)), a 2-player grid game; n-player Nim extensions (Li (1978)); poisoned chocolate bar games (Robin (1989)); Greedy Nim (lbert and Nowakowski (2004)), where players choose the largest heap; and k-bounded Greedy Nim (Lv et al. (2018)), which restricts the number of tokens that can be removed from a heap per turn. Recent works have introduced further variants of Nim. Duchêne et al. (2016) proposed a two-stage version where players first build stacks by placing tokens, then play Nim on the resulting configuration. Xu and Zhu (2018) introduced Bounded Greedy Nim, combining constraints from both Bounded and Greedy Nim, and provided a complete solution. Van den Bergh et al. (2022) studied Nim under imperfect information, computing Nash equilibria for several setups.

4. THE GR PHS STUDIED, ND THE M IN RESULTS OF THIS P PER

We begin by listing all distinct unlabeled graphs comprising 4 edges with no isolated vertices (i.e. a vertex with no neighbour), as for a graph with an isolated vertex, our game on such a graph reduces to the analysis on the corresponding subgraph with all isolated vertices removed. This, in turn, implies that the number of vertices is between 4 and 8 vertices (i.e. where $|E| = 4$ and $4 \leqslant |V| \leqslant 8$).

lthough we have stated above that it suffices to consider all unlabeled distinct graphs, the vertices of each graph in Figures 2 and 3 have been labeled for ease of exposition (when we state our results pertaining to these graphs). We exclude all graphs that contain at least one isolated vertex, since such graphs can be reduced to graphs with smaller number of vertices by eliminating all isolated vertices anyway. Likewise, in the entirety of this paper, we assume that each edge of the graph under consideration receives an initial weight that is a *strictly* positive integer, i.e. each initial weight configuration we consider is such that no edge of the graph we are concerned with is left with weight 0. This is an important assumption that is to be borne in mind while reading and interpreting the main results.

The graphs that we study in this paper have all been illustrated in Figure 2. Of these, the graphs G_1, H_2, H_3, I_1 and I_2 are all galaxy graphs, and the characterization the winning and losing initial weight configurations on these graphs follows directly from Theorem 2.3. Moreover, the result characterizing the winning and losing initial weight configurations on F_1 was proved in Williams (2017), and we have included it in our paper for the sake of completeness.

Theorem 4.1. *We enumerate here our results concerning all of the galaxy graphs illustrated in Figure 2:*

(i) Every *initial weight configuration on G_1, with the sum of all edge-weights being strictly positive, is winning.*

(ii) n initial weight configuration $w_0\ B), w_0\ BC), w_0\ DE), w_0\ EF))$ *on H_2 is losing if and only if* $w_0\ B) + w_0\ BC) = w_0\ DE) + w_0\ EF)$.

(iii) n initial weight configuration $w_0\ B), w_0\ C), w_0\ D), w_0\ EF))$ *on H_3 is losing if and only if* $w_0\ B) + w_0\ C) + w_0\ D) = w_0\ EF)$.

(iv) n initial weight configuration $w_0\ B), w_0\ BC), w_0\ DE), w_0\ FG))$ *on I_1 is losing if and only if the triple* $w_0\ B) + w_0\ BC), w_0\ DE), w_0\ FG))$ *is balanced.*

(v) n initial weight configuration $w_0\ B), w_0\ CD), w_0\ EF), w_0\ GH))$ *on I_2 is losing if and only if the tuple* $w_0\ B), w_0\ CD), w_0\ EF), w_0\ GH))$ *is balanced.*

Theorem 4.2 (Theorem 2 of Williams (2017)). *n initial weight configuration* $w_0\ B), w_0\ BC), w_0\ CD), w_0\ D\ ))$ *on the graph F_1, illustrated in Figure 2, is losing if and only if* $w_0\ B) = w_0\ CD)$ *and* $w_0\ BC) = w_0\ D\ )$.

Having stated Theorem 4.1 and Theorem 4.2, we now come to the statements of the main results of this paper, most of which necessitate far more involved proofs. Before we state these, we include here Figure 3, which illustrate, for the reader's convenience, only those graphs (namely, F_2, G_2, G_3, G_4 and H_1) with 4 edges, and up to 8 vertices, that are neither galaxy graphs nor those already explored in Williams (2017). We continue by first stating our results pertaining to the graphs F_2, G_2 and G_3, as follows:

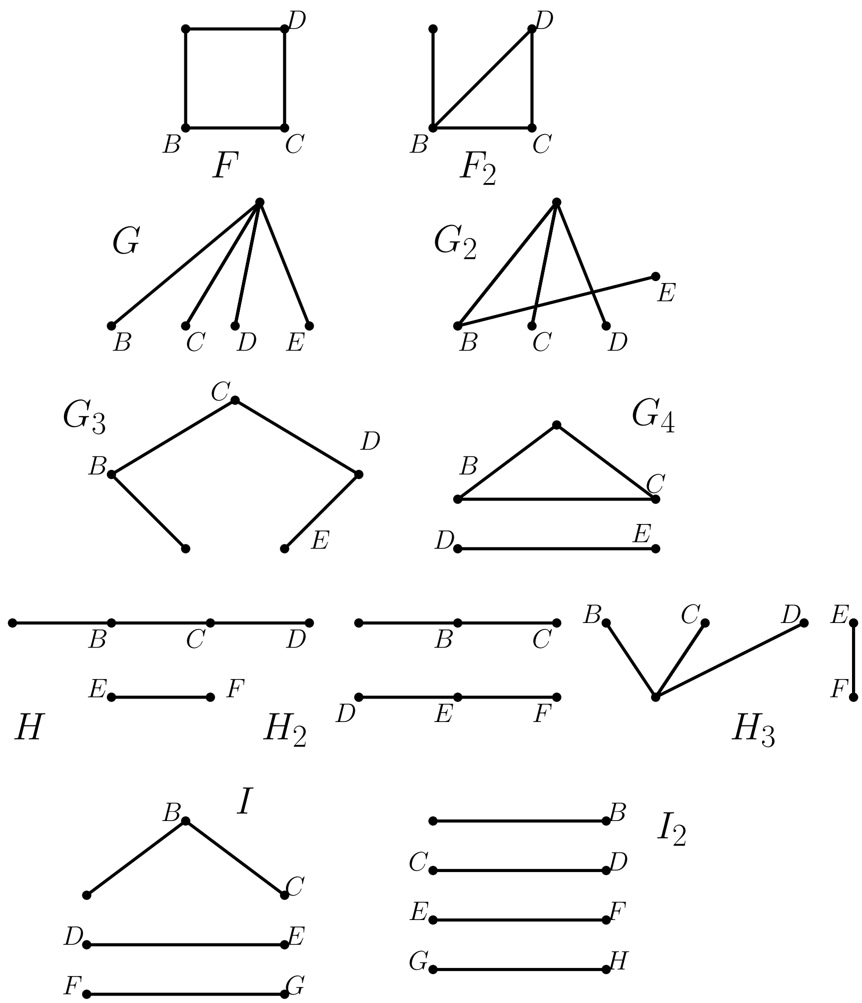

FIGURE 2. ll graphs with precisely 4 edges and no isolated vertices

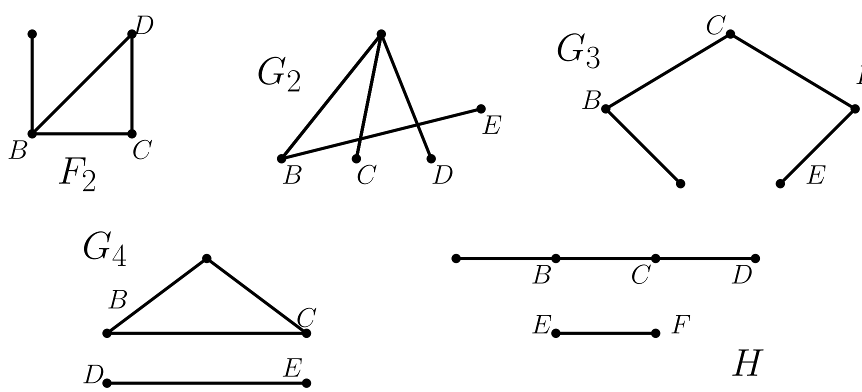

FIGURE 3. The main non-galaxy graphs studied in this paper, with precisely 4 edges and at most 8 vertices

Theorem 4.3. *n initial weight configuration* $w_0 \quad B), w_0\ BC), w_0\ CD), w_0\ DB))$ *on the graph* F_2*, illustrated in Figure 3, is losing if and only if* $w_0\ BC) = w_0\ DB)$ *and* $w_0\ CD) = w_0 \quad B) + w_0\ BC)$.

Theorem 4.4. Every *initial weight configuration on each of the graphs* G_2 *and* G_3*, illustrated in Figure 3, is winning.*

The two graphs that prove to be the most challenging when it comes to the task of characterizing the winning and losing initial weight configurations on them, are G_4 and H_1. We dedicate §4.1 to the statements of all of our results pertaining to the graph G_4, and we dedicate §4.2 to the statements of all of our results pertaining to the graph H_1. While we have been able to provide a complete characterization of the winning and losing initial weight configurations on G_4, our findings regarding H_1 provide a partial description of the winning and losing initial weight configurations on H_1.

4.1. **complete characterization of the winning and losing configurations on** G_4**.** We begin §4.1 with some discussions on a couple of notions, as these are going to aid us in stating our results pertaining to the graph G_4 in a more concise manner than otherwise possible. We recall for the reader that by a *multiset*, we mean an unordered collection of elements with repetitions allowed (in other words, a multiset would be reduced to a set if precisely one copy of each distinct element is retained). With a slight abuse of notation, which is nonetheless quite common in the literature, we express a multiset in the same manner as a set: by listing its (not necessarily distinct) elements within curly brackets. For example, $\{a,a,a,b,b,c\}$ represents a multiset in which the element a appears three times, the element b appears twice, and the element c appears once. Given any element belonging to a multiset, the number of copies of this element that appear in that multiset is referred to as its *repetition number*. Thus, the repetition number of a in the multiset $\{a,a,a,b,b,c\}$ equals 3, that of b equals 2 and that of c equals 1. We emphasize here that the ordering of the elements in a multiset is irrelevant: for instance, continuing with the same example as in the previous sentence, $\{a,a,a,b,b,c\}$ and $\{c,a,b,a,b,a\}$ represent the same multiset. The *set obtained from a multiset* is simply the (unordered) collection of all distinct elements appearing in that multiset – thus, the set obtained from the multiset $\{a,a,a,b,b,c\}$ is, simply, $\{a,b,c\}$. Two multisets are defined to be equal if and only if

(i) the sets obtained from these two multisets are the same,
(ii) for each distinct element, its repetition number in one multiset equals its repetition number in the other.

Next, we introduce the following definition:

Definition 4.5. Given $k,m,i \in \quad$ and $\ell \in \quad_0$ such that

$$i \in \{1,2, \quad ,m+1\} \quad \text{and} \quad \frac{m\ m+1)}{2} \leqslant k \leqslant \frac{m\ m+3)}{2}, \tag{4.1}$$

we say that a multiset S, consisting of 3 elements in total, is $\ k,\ell,m,i)$*-special* if S, as a multiset, equals $\{k+1+\ m+1)\ell, k+i+\ m+1)\ell, k+m+2 \quad i+\ m+1)\ell\}$. We call a multiset S simply *special* if S is $\ k,\ell,m,i)$-special for *some* $k,m,i \in \quad$ and $\ell \in \quad_0$ satisfying (4.1).

Some examples may help elucidate this definition further. If m is even, and $i \in \{2, \quad ,m/2, m/2+2, \quad ,m+1\}$, then the elements $k+1+\ m+1)\ell$, $k+i+\ m+1)\ell$ and $k+m+2 \quad i+\ m+1)\ell$ are all distinct, and the multiset S containing these elements becomes simply a set; on the other hand, for m even, we have $k+1+\ m+1)\ell = k+i+\ m+1)\ell$ if $i=1$ and we have $k+i+\ m+1)\ell = k+m+2 \quad i+\ m+1)\ell$ if $i = m/2+1$, so that the multiset S either becomes $\{k+1+\ m+1)\ell, k+1+\ m+1)\ell, k+m+1+\ m+1)\ell\}$, or it becomes $\{k+1+\ m+1)\ell, k+m+1+\ m+1)\ell, k+m+1+\ m+1)\ell\}$. If m is odd, and $i>1$, once again the multiset S becomes simply a set, whereas if $i=1$, the multiset S becomes $\{k+1+\ m+1)\ell, k+1+\ m+1)\ell, k+m+1+\ m+1)\ell\}$.

It is worthwhile to note here that, since $m+1)\ m+2)/2\quad 1 = m\ m+3)/2$ for each $m \in$, for each $k \in$, there exists a *unique* $m \in$ such that (4.1) holds, and the sets $\{k \in\ : m\ m+1) \leqslant 2k \leqslant m\ m+3)\}$, for all $m \in$, yields a *partition* of the set .

Our result, characterizing *fully* the winning and losing initial weight configurations on G_4, is as follows:

Theorem 4.6. *n initial weight configuration* $w_0\quad B), w_0\ BC), w_0\ C\quad), w_0\ DE))$ *on* G_4 *is losing if and only if precisely one of the following two conditions holds:*

- (1) *letting* $w_0\ DE) = k$, *there exist* $m, i \in$ *and* $\ell \in\ _0$ *such that* (4.1) *is satisfied and* $\{w_0\quad B), w_0\ BC), w_0\ C\quad)\}$*, as a multiset, is* $k, \ell, m, i)$*-special;*
- (2) $w_0\ DE) = w_0\quad B) + w_0\ BC) + w_0\ C\quad)$*, the edge-weights* $w_0\quad B)$*,* $w_0\ BC)$ *and* $w_0\ C\quad)$ *are not all equal, and the multiset* $\{w_0\quad B), w_0\ BC), w_0\ C\quad)\}$ *is* not *special.*

4.2. **partial characterization of the winning and losing configurations on H_1.** The most challenging of all the graphs we have dealt with in this paper is H_1, and we are able to obtain only a partial characterization of the winning and losing initial weight configurations on H_1. The nature of our findings itself speaks for the difficulty that one faces as one attempts to generalize the patterns we have observed so far, and we include a brief discussion following Theorem 4.10 to further reinforce our belief that a *full* characterization of the winning and losing configurations on H_1 would be a commendable, and possibly quite arduously accomplished, feat. Since our findings encompass *several* different possible scenarios, we have included, for the convenience of the reader, a flowchart that summarizes these findings, in Figure 4 at the end of §4.2.

We begin by enumerating a few results pertaining to the winning initial weight configurations on H_1:

Lemma 4.7. *n initial weight configuration on* H_1 *is winning whenever it satisfies at least one of*

$$w_0\quad B) \leqslant w_0\ EF) \leqslant w_0\quad B) + w_0\ BC), \tag{4.2}$$

$$w_0\ CD) \leqslant w_0\ EF) \leqslant w_0\ BC) + w_0\ CD) \tag{4.3}$$

less obvious criterion that guarantees that a weight configuration on H_1 is winning is captured by our next result. Before we state it, we recall, for the reader, that any base-2 representation of $x \in$ is written as $x = \ a_r a_{r\ 1}\quad a_1 a_0)_2$, so that $x = \ {}^{r}_{i=0} a_i 2^i$, and wherever needed, we allow for a_r to be equal to 0 (for instance, $7 = \ 111)_2$, but if needed, we may also write $7 = \ 0111)_2$).

Theorem 4.8. *Let* $w_0\ EF) = \ a_s a_{s\ 1}\quad a_1 a_0)_2$, $w_0\quad B) = \ b_s b_{s\ 1}\quad b_1 b_0)_2$ *and* $w_0\ CD) = \ c_s c_{s\ 1}\quad c_1 c_0)_2$*, where s is chosen such that at least one of* a_s*,* b_s *and* c_s *equals* 1*. We define I to be the maximum element of the set S, where* $S = \{i \in \{0, 1,\quad , s\} : a_i + b_i + c_i$ *is odd*$\}$*, provided S is non-empty. The initial weight configuration* $w_0\quad B), w_0\ BC), w_0\ CD), w_0\ EF))$ *is winning whenever*

- *(B1)* $w_0\ BC) \geqslant 1$ *and S is empty,*
- *(B2) or* $w_0\ BC) \geqslant 0$ *and S is non-empty and at least one of* b_I *and* c_I *equals* 1.

Before we proceed with our next results, we set down the following notation which will be used repeatedly in the sequel: given $k \in$, we let $f\ k)$ denote the unique non-negative integer that satisfies the inequalities $2^{f\ k)} \leqslant k < 2^{f\ k)+1}$.

Lemma 4.9. *ny initial weight configuration* $w_0\quad B), w_0\ BC), w_0\ CD), w_0\ EF))$ *on* H_1 *is winning as long as* $w_0\ EF) = k$*,* $w_0\quad B) = 2^{f\ k)+1} m_1 + \ell_1$ *and* $w_0\ CD) = 2^{f\ k)+1} m_2 + \ell_2$*, for any* $k \in$ *, any* $m_1, m_2 \in\ _0$*, and any* $\ell_1, \ell_2 \in \{0, 1,\quad , 2^{f\ k)+1}\quad 1\}$*, such that (i) either* $m_1 \neq m_2$ *(ii) or* $\min\{\ell_1, \ell_2\} \geqslant k$ *(iii) or* $k \in \{\ell_1, \ell_2\}$ *and either* $\min\{\ell_1, \ell_2\} > 0$ *or* $w_0\ BC) = 0$*.*

So far, we have only enumerated results pertaining to winning configurations on H_1. Our next result is a lengthy one, summarizing the various losing configurations on H_1 that we have, so far, been able to identify:

Theorem 4.10. *In what follows, we set $w_0 \; EF) = k$ for some $k \in \quad$, and we let $m \in \quad_0$ (making sure that none of the initial edge-weights equals 0). Each of the following is true:*

(i) *ny initial weight configuration on H_1 that is of the form*

$$\{w_0 \quad B), w_0 \; CD)\} = \{2^{f \; k)+1}m + r, 2^{f \; k)+1}m + k \quad r \quad s\} \quad \text{and} \quad w_0 \; BC) = s, \tag{4.4}$$

for $r \in \{0,1,3\}$ and $s \in \{1,2, \quad ,k \quad 2r\}$, is losing on H_1.

(ii) *ny initial weight configuration on H_1 that is of the form*

$$\{w_0 \quad B), w_0 \; CD)\} = \{2^{f \; k)+1}m + r, 2^{f \; k)+1}m + k + r \quad s\} \quad \text{and} \quad w_0 \; BC) = s, \tag{4.5}$$

with $r \in \{2,4\}$, $s \in \{1, \quad ,r \quad 1\}$ and $k \equiv j \bmod 2^{f \; r)+1}$ for some $j \in \{s, s+1, \quad ,r \quad 1\}$, is losing on H_1.

(iii) *ny initial weight configuration on H_1 that is of the form*

$$\{w_0 \quad B), w_0 \; CD)\} = \{2^{f \; k)+1}m + r, 2^{f \; k)+1}m + k \quad r \quad s\} \quad \text{and} \quad w_0 \; BC) = s, \tag{4.6}$$

with $r \in \{2,4\}$, $s \in \{1, \quad ,r \quad 1\}$ and $k \geqslant 3r$ with $k \equiv j \bmod 2^{f \; r)+1}$ for some $j \in \{0,1, \quad ,2^{f \; r)+1} \quad 1\} \setminus \{s, s+1, \quad ,r \quad 1\}$, is losing on H_1.

(iv) *ny initial weight configuration on H_1 that is of the form*

$$\{w_0 \quad B), w_0 \; CD)\} = \{2^{f \; k)+1}m + r, 2^{f \; k)+1}m + k \quad r \quad s\} \quad \text{and} \quad w_0 \; BC) = s, \tag{4.7}$$

with $r \in \{2,4\}$, $s \in \{r, r+1, \quad ,k \quad 2r\}$ and $k \geqslant 3r$, is losing on H_1.

Theorem 4.10 does seem to reveal a pattern – however, this pattern does not seem to extend any further. For instance, as we explored configurations on H_1 that are of the form

$$w_0 \quad B) = 2^{f \; k)+1}m + 5, \; w_0 \; BC) = s, \; w_0 \; CD) = 2^{f \; k)+1}m + t, \; w_0 \; EF) = k,$$

for $t \geqslant 5$ and $s \geqslant 1$, we found that setting $s = 1$, $t = 6$ and $k = 11$ yields a losing configuration, even though this does not conform to the patterns suggested by Theorem 4.10. Despite repeated attempts, we have not been able to extend Theorem 4.10 to encompass a broader class of losing configurations on H_1.

We include here two small results, of which Lemma 4.11 is a corollary of our claim, stated in Theorem 4.10, that any configuration on H_1 of the form given by (4.4), with $r = 0$, is losing, whereas Lemma 4.13 is a corollary of Theorem 2.1 and Theorem 2.3 – the motivation for stating these lies in being able to directly use their conclusions in various steps of the proof of Theorem 4.10.

Lemma 4.11. *n initial weight configuration on H_1 is winning whenever we have $w_0 \; EF) = k$, $w_0 \quad B) = 2^{f \; k)+1}m_1 + \ell_1$, $w_0 \; CD) = 2^{f \; k)+1}m_2 + \ell_2$ and $w_0 \; BC) > k \quad \min\{\ell_1, \ell_2\}$, for any $k \in \quad$, any $m_1, m_2 \in \quad_0$, and any $\ell_1, \ell_2 \in \{0,1, \quad ,2^{f \; k)+1} \quad 1\}$ with $\min\{\ell_1, \ell_2\} < k$. Moreover, any initial weight configuration on H_1 with $w_0 \; BC) > w_0 \; EF)$ is also winning.*

Remark 4.12. If $w_0 \; EF) = 0$, then the corresponding initial weight configuration on H_1 is winning. This can be argued as follows: assuming, without loss of generality, that $w_0 \quad B) \geqslant w_0 \; CD)$, P_1 removes, in the first round, weight $w_0 \; CD) \quad w_0 \quad B)$ from the edge CD, and the entire edge BC, leaving P_2 with a galaxy graph consisting of the edges $\quad B$ and CD, with $w_1 \quad B) = w_1 \; CD)$. Consequently, P_2 loses by Theorem 2.3.

Because of the last assertion made in Remark 4.12, we need not consider, in the proof of Theorem 4.10, the scenario where P_1, in the first round, removes the entire edge EF (since, in such cases, we already know that P_2 would win).

Lemma 4.13. *Consider a galaxy graph consisting of three pairwise-vertex-disjoint star graphs, with the sum of the initial edge-weights assigned to the edges of the i-th star graph being equal to w_i, for $i \in \{1,2,3\}$. If $w_3 = k$ for some $k \in \quad$, and $w_i = 2^{f \; k)+1}m + \ell_i$ for some $m \in \quad_0$ and $\ell_i \in \{0,1, \quad ,2^{f \; k)+1} \quad 1\}$ for each $i \in \{1,2\}$, then this configuration is winning whenever $\ell_1 + \ell_2 < k$.*

s mentioned at the start of §4.2, for a quick summary of our main results regarding the winning and losing initial weight configurations on the graph H_1, we urge the reader to vide Figure 4.

5. PROOF OF OUR RESULTS REG RDING THE GR PHS F_2, G_2 ND G_3

Before we begin with the proofs of our results, we state here a result from Williams (2017) that is going to be of use to us in the sequel:

Theorem 5.1 (Page 13 of Williams (2017)). *When G is a triangle (i.e. $|V| = 3$ and $|E| = 3$), an initial weight configuration on G is losing if and only if the edge-weight assigned to each of the three edges in G is the same.*

During the analysis of a Game of Graph Nim played on a given graph $G = \ V,E)$, we denote by $w_i\ e)$, for each $e \in E$, the edge-weight remaining on the edge e *after* the i-th round of the game has been completed, for each $i \in \ $.

Proof of Theorem 4.3. In order to establish Theorem 4.3, we begin by showing that whenever an initial weight configuration $\ w_0\ \ B), w_0\ BC), w_0\ CD), w_0\ DB))$, on F_2, satisfies the criteria:

$$w_0\ BC) = w_0\ DB) \text{ and } w_0\ CD) = w_0\ \ B) + w_0\ BC), \tag{5.1}$$

it is losing. The proof of this claim happens via induction on $w_0\ CD)$, the base case for which has been addressed in §8.1 of §8.

Suppose we have shown, for some $K \in \ $ with $K \geqslant 2$, that whenever $\ w_0\ \ B), w_0\ BC), w_0\ CD), w_0\ DB))$ satisfies (5.1) along with the constraint that $w_0\ CD) = k$ for some $k \leqslant K$, it is a losing configuration. We now consider the configuration

$$w_0\ BC) = w_0\ DB) = i, \quad w_0\ \ B) = \ K+1\ \ i) \quad \text{and} \quad w_0\ CD) = \ K+1), \tag{5.2}$$

for *any* $i \in \ $ with $i \leqslant K$. The first round of the Game of Graph Nim played on this initial configuration can unfold in one of the following ways:

(i) Suppose P_1 selects a vertex such that the edge-weights of the edges belonging to some subset of $\{\ \ B, DB, BC\}$ are modified, while all else remain intact. We then have

$$w_1\ BC) = j_1, \quad w_1\ DB) = j_2 \quad \text{and} \quad w_1\ \ B) = j_3,$$

where $\max\{j_1, j_2\} \leqslant i$ and $j_3 \leqslant\ K+1\ \ i)$, with at least one of these three inequalities being strict, or equivalently, either $j_1 + j_3 \leqslant K$ or $j_2 + j_3 \leqslant K$. Without loss of generality, we assume that $j_1 \leqslant j_2$ – which, in turn, ensures that we must have $j_1 + j_3 \leqslant K$. Then P_2 chooses the vertex D and removes weight $\ j_2\ \ j_1)$ from DB and weight $\ K+1)\ \ \ j_1+j_3)$ from CD in the second round, leaving P_1 with

$$w_2\ BC) = w_2\ DB) = j_1, \quad w_2\ \ B) = j_3 \quad \text{and} \quad w_2\ CD) = j_1 + j_3,$$

and as $j_1 + j_3 \leqslant K$, we know that P_1 loses by our induction hypothesis.

(ii) Suppose P_1 chooses a vertex such that the edge-weights of CD and BC (analogously, CD and DB) are modified, while all else remain intact. We then have

$$w_1\ BC) = j, \quad w_1\ DB) = i, \quad w_1\ \ B) = \ K+1\ \ i) \quad \text{and} \quad w_1\ CD) = k,$$

for some $j \leqslant i$ and some $k \leqslant K$ (it suffices to consider the case where $k \leqslant K$, since the case of $k = \ K+1)$ is already contained in the previous scenario). There are a few subcases to consider here:

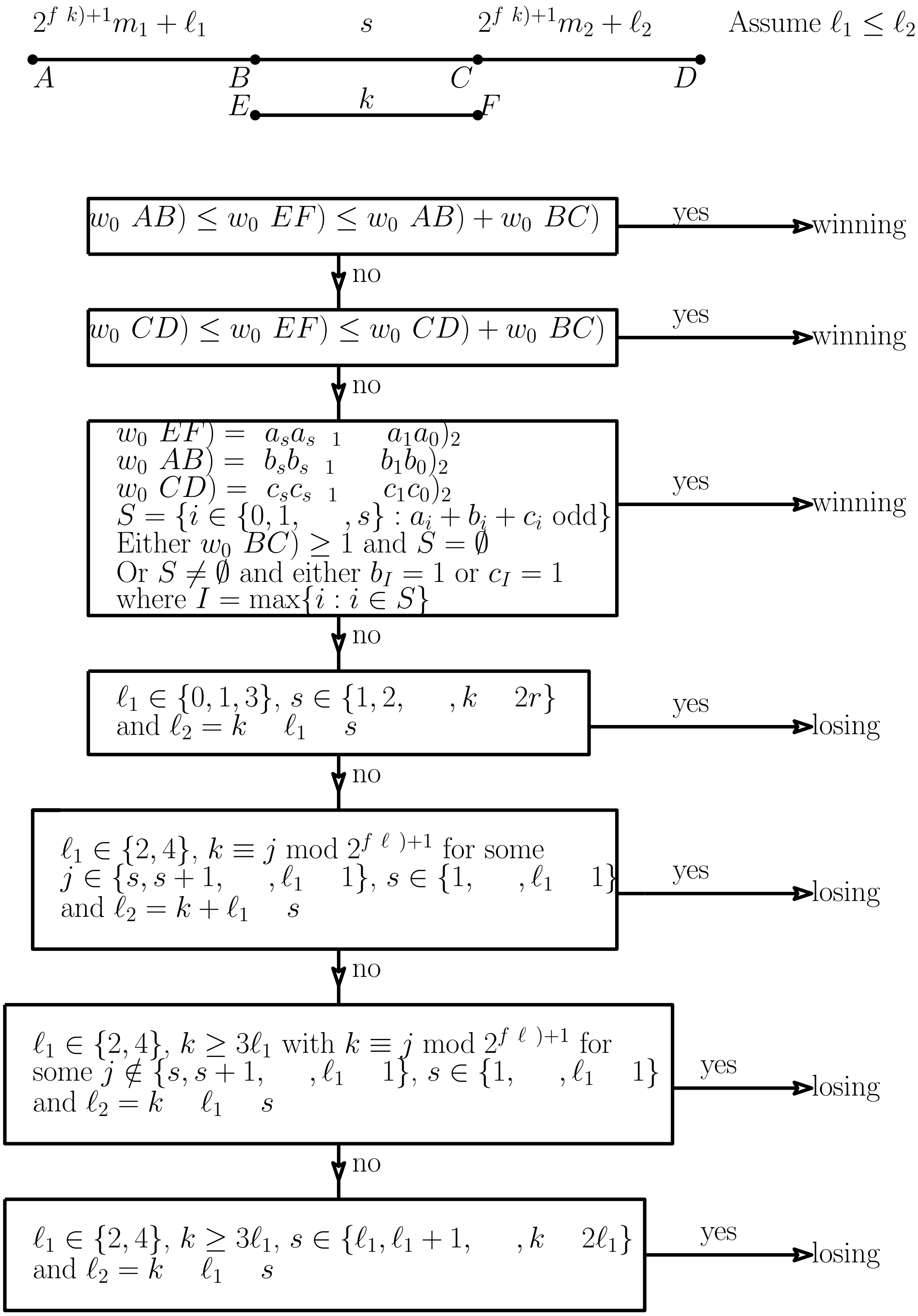

FIGURE 4. flowchart summarizing our findings regarding H_1

(a) When $K+1) \quad k \geqslant i \quad j$, then P_2 selects the vertex B and removes weight $i \quad j)$ from DB and $K+1 \quad i) \quad k \quad j)$ from B in the second round, leaving P_1 with

$$w_2\ BC) = w_2\ DB) = j, \quad w_2 \quad B) = k \quad j \quad \text{and} \quad w_2\ CD) = k,$$

and as $k \leqslant K$, we know that P_1 loses by our induction hypothesis.

(b) When $K+1) \quad k < i \quad j$, then P_2 selects the vertex D and removes weight $i \quad j)$ from DB and weight $k \quad [\{ \ K+1) \quad i\} + j]$ from CD in the second round, leaving P_1 with

$$w_2\ BC) = w_2\ DB) = j, \quad w_2 \quad B) = \ K+1) \quad i \quad \text{and} \quad w_2\ CD) = \ K+1) \quad i+j,$$

and since $K+1) \quad i+j < k \leqslant K$, we know that P_1 loses by our induction hypothesis.

This completes the proof of our claim that the initial configuration in (5.2) is losing, and thereby completes the inductive proof of our claim that the initial configuration in (5.1) is losing. We are now left to show that any initial configuration on F_2 that does *not* satisfy (5.1) is winning, which we argue as follows.

ny initial configuration $w_0 \quad B), w_0\ BC), w_0\ CD), w_0\ DB))$ on F_2 not satisfying (5.1) must either be such that $w_0\ BC) \neq w_0\ DB)$ or $w_0\ CD) \neq w_0\ BC) + w_0 \quad B)$. We can, thus, consider the following two possibilities:

(i) Suppose $w_0\ BC) \neq w_0\ DB)$, and without loss of generality, we assume that $w_0\ BC) > w_0\ DB)$. Let us set

$$w_0\ DB) = i, \quad w_0\ BC) = j, \quad w_0\ CD) = k \quad \text{and} \quad w_0 \quad B) = \ell$$

If $k \geqslant i+\ell$, then P_1 selects the vertex C and removes weight $j \quad i)$ from BC and weight $k \quad i+\ell)$ from CD in the first round, leaving P_2 with

$$w_1\ DB) = w_1\ BC) = i, \quad w_1 \quad B) = \ell \quad \text{and} \quad w_1\ CD) = i+\ell,$$

which is of the form given by (5.1), so that P_2 loses. If, on the other hand, $k < i+\ell$, then P_1 selects the vertex B and removes weight $j \quad i)$ from BC and weight $\ell \quad k \quad i)$ from B in the first round, leaving P_2 with

$$w_1\ DB) = w_1\ BC) = i, \quad w_1 \quad B) = k \quad i \quad \text{and} \quad w_1\ CD) = k,$$

which is, once again, of the form given by (5.1), so that P_2 loses.

(ii) Suppose, now, that $w_0\ BC) = w_0\ DB)$, but $w_0 \quad B) + w_0\ BC) \neq w_0\ CD)$. Once again, letting

$$w_0\ DB) = w_0\ BC) = i, \quad w_0\ CD) = k \quad \text{and} \quad w_0 \quad B) = \ell,$$

we consider two subcases: (a) when $k > i+\ell$, P_1 removes weight $k \quad i+\ell)$ from CD in the first round, (b) whereas when $k < i+\ell$, P_1 removes weight $\ell \quad k \quad i)$ from B in the first round, so that, in either scenario, P_2 is left with a configuration of the form given by (5.1), leading to her defeat.

This completes the proof of our claim that any initial weight configuration on F_2 that does not satisfy (5.1) is winning. This also concludes the proof of Theorem 4.3.

Proof of Theorem 4.4. In order to establish our claim regarding the graph G_2, we consider the following cases:

(i) If $w_0\ BE) \geqslant w_0 \quad C) + w_0 \quad D)$, then P_1 selects the vertex B, removes the entire edge B, and removes weight $w_0\ BE) \quad \{w_0 \quad C) + w_0 \quad D)\}$ from the edge BE, so that P_2 is left with the galaxy graph consisting of the edges BE, C and D, with $w_1\ BE) = w_1 \quad C) + w_1 \quad D) = w_0 \quad C) + w_0 \quad D)$. Therefore, P_2 loses by Theorem 2.3.

(ii) If $w_0\ BE) < w_0 \quad C) + w_0 \quad D)$, then P_1 selects the vertex , removes the entire edge B, and removes from at least one of C and D sufficient weight such that $w_1 \quad C) + w_1 \quad D) = w_0\ BE) = w_1\ BE)$ (the choice of weights to be removed from C and / or D for this equality to hold need not be unique), so that, once again, P_2 loses by Theorem 2.3.

This completes the proof of our claim that *all* initial weight configurations on G_2 are winning. We now come to the graph G_3, and we consider the following cases:

(i) Suppose $w_0\ DE) \geqslant w_0\ \ B) + w_0\ BC)$ (an analogous situation would be where $w_0\ \ B) \geqslant w_0\ CD) + w_0\ DE)$). In this case, P_1 selects the vertex D, removes the entire edge CD, and removes weight $w_0\ DE)\ \ \{w_0\ \ B) + w_0\ BC)\}$ from the edge DE in the first round, leaving P_2 with a galaxy graph consisting of the edges $\ B$, BC and DE, where $w_1\ DE) = w_0\ \ B) + w_0\ BC) = w_1\ \ B) + w_1\ BC)$. Therefore, P_2 loses by Theorem 2.3.

(ii) Suppose $w_0\ \ B) + w_0\ BC) > w_0\ DE) \geqslant w_0\ \ B)$ (an analogous situation would be where $w_0\ CD) + w_0\ DE) > w_0\ \ B) \geqslant w_0\ DE)$). In this case, P_1 selects the vertex C, removes the entire edge CD, and removes weight $w_0\ \ B) + w_0\ BC)\ \ w_0\ DE)$ from the edge BC. Note, here, that the inequalities $w_0\ \ B) + w_0\ BC) > w_0\ DE) \geqslant w_0\ \ B)$ ensures that the weight removed from BC is strictly positive and bounded above by $w_0\ BC)$. This leaves P_2 with a galaxy graph consisting of the edges $\ B$, BC and DE, where $w_1\ BC) = w_0\ BC)\ \ \{w_0\ \ B) + w_0\ BC)\ \ w_0\ DE)\} = w_0\ DE)\ \ w_0\ \ B) = w_1\ DE)\ \ w_1\ \ B)$, and P_2 loses by Theorem 2.3.

(iii) The last remaining case to consider is where $w_0\ \ B) + w_0\ BC) > w_0\ \ B) > w_0\ DE)$. But here, if $w_0\ \ B) \geqslant w_0\ CD) + w_0\ DE)$, then this becomes analogous to the first scenario considered above, and if $w_0\ \ B) < w_0\ CD) + w_0\ DE)$, then this becomes analogous to the second scenario considered above. Either way, P_1 wins.

This completes the proof of our claim that *all* initial weight configurations on G_3 are winning.

6. PROOFS OF OUR RESULTS PERT INING TO THE GR PH G_4

We establish Theorem 4.6 via the proofs of several results that have been enumerated below as a list for the reader's convenience:

(D1) n initial weight configuration $\ w_0\ \ B), w_0\ BC), w_0\ C\ \ ), w_0\ DE))$ on G_4 is winning whenever it satisfies the inequalities $\min\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\} \leqslant w_0\ DE) \leqslant w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )\ \ \min\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$.

(D2) n initial weight configuration $\ w_0\ \ B), w_0\ BC), w_0\ C\ \ ), w_0\ DE))$ on the graph G_4 is losing whenever the criterion stated in (1) is satisfied.

(D3) If $0 < w_0\ DE) < \min\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ but the weight configuration does not satisfy the criterion stated in part (1) of Theorem 4.6, then P_1 wins.

(D4) ny initial weight configuration $\ w_0\ \ B), w_0\ BC), w_0\ C\ \ ), w_0\ DE))$ on the graph G_4, that satisfies the criterion stated in part (2) of Theorem 4.6, is losing.

(D5) If $\ w_0\ DE) > w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )\ \ \min\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$, but $w_0\ DE) \neq w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )$, and the multiset $\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ is *not* special, the initial weight configuration $\ w_0\ \ B), w_0\ BC), w_0\ C\ \ ), w_0\ DE))$ is winning on G_4.

It is fairly straightforward to see that, once each of (D1), (D2), (D3), (D4) and (D5) has been proved, the proof of Theorem 4.6 would be complete.

6.1. **Proof of Step D1 for proving Theorem 4.6.** Without loss of generality (since the edges $\ B$, BC and $C\ \ $ play symmetric roles in the graph G_4), let us assume that $w_0\ \ B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$, so that the inequalities stated in the hypothesis of (D1) boil down to: $w_0\ \ B) \leqslant w_0\ DE) \leqslant w_0\ BC) + w_0\ C\ \ )$. We split the analysis into two cases.

If $w_0\ \ B) \leqslant w_0\ DE) \leqslant w_0\ \ B) + w_0\ C\ \ )$, then P_1 selects the vertex C, removes the entire edge BC, and removes weight $w_0\ \ B) + w_0\ C\ \ )\ \ w_0\ DE)$ from the edge $C\ \ $. This leaves P_2 with a galaxy graph comprising two components: the first of which consists of the edges $\ B$ and $C\ \ $, the second of which consists of only the edge DE, and $w_1\ C\ \ ) = w_0\ C\ \ )\ \ \{w_0\ \ B) + w_0\ C\ \ )\ \ w_0\ DE)\} = w_0\ DE)\ \ w_0\ \ B) = w_1\ DE)\ \ w_1\ \ B)$. Consequently, P_2 loses by Theorem 2.3.

If $w_0(AB)+w_0(CA) < w_0(DE) \leqslant w_0(BC)+w_0(CA)$, then P_1 removes weight $w_0(BC)+w_0(CA)-w_0(DE)$ from the edge BC, and the entire edge AB, in the first round, leaving P_2 with a galaxy graph comprising two components: one is a star graph with the edges BC and CA, the other is the single edge DE, such that $w_1(BC)+w_1(CA) = w_0(DE) = w_1(DE)$. Once again, P_2 loses by Theorem 2.3.

6.2. **Proof of Step D2 for proving Theorem 4.6.** The proof of Step (D2) is accomplished via induction, which happens with respect to the parameters m, k and ℓ involved in the statement of (A1). The base case, corresponding to $m=k=1$ and all $\ell \in \mathbb{N}_0$, has been proved in §8.2 of §8. Suppose, now, that for some $M \in \mathbb{N}$, some $K \in \mathbb{N}$ satisfying the inequalities

$$\frac{(M+1)(M+2)}{2} \leqslant K \leqslant \frac{(M+1)(M+4)}{2}, \tag{6.1}$$

and some $L \in \mathbb{N}_0$, we have already shown that

- (I1) each configuration satisfying the hypothesis of (A1), with $m \leqslant M$ and m, k and i satisfying (4.1), is losing;
- (I2) each configuration satisfying the hypothesis of (A1), with $m = M+1$, all $k \in \mathbb{N}$ such that $(M+1)(M+2)/2 \leqslant k < K$, all $i \in \{1,2,\ldots,M+2\}$, and all $\ell \in \mathbb{N}_0$, is losing;
- (I3) the configuration satisfying the hypothesis of (A1), with $k=K$, $m=M+1$, all $i \in \{1,2,\ldots,m+1\}$, and all $\ell \in \mathbb{N}_0$ with $\ell < L$, is losing.

Note that the induction hypothesis stated in (I3) becomes vacuous (i.e. no longer needs to be considered) when $L=0$. In the proof that follows (of showing that the configuration given by (6.2) is losing), we incorporate the possibility of $L=0$. In other words, we do not separately prove that (6.2) is losing for $L=0$ – rather, we show that the proof for $L=0$ is a subset of the proof for $L \in \mathbb{N}$, obtained by simply *not* considering the case where P_1 removes weights from at most two of AB, BC and CA in the first round such that $K+1+(M+2)L > \min\{w_1(AB), w_1(BC), w_1(CA)\} > K$.

Likewise, the induction hypothesis in (I2) becomes vacuous when $K=(M+1)(M+2)/2$ – this, too, has been taken into account in the proof that follows, and has not been treated separately. Specifically, in this case, if, after the second round, the edge-weight of DE decreases strictly from what it was at the beginning, i.e. $w_2(DE)=k$ for some $k<K$, we have $k \leqslant (M+1)(M+2)/2-1 = M(M+3)/2$. Consequently, k satisfies (4.1) for some $m \leqslant M$, and hence, the induction hypothesis in (I1) can be applied.

We now come to the inductive step of our proof, and focus on the initial weight configuration

$$\begin{aligned} & w_0(AB) = K+1+(M+2)L,\ w_0(BC) = K+i+(M+2)L, \\ & w_0(CA) = K+M+3-i+(M+2)L,\ w_0(DE) = K, \end{aligned} \tag{6.2}$$

where $i \in \{1,2,\ldots,M+2\}$. Here, we have assumed, without any loss of generality, that $w_0(AB) \leqslant w_0(BC) \leqslant w_0(CA)$, which yields $2i \leqslant (M+3)$. We consider the first round of the game played on this configuration:

(i) Suppose P_1 removes some weight from at most two of the edges AB, BC and CA in the first round, such that $\min\{w_1(AB), w_1(BC), w_1(CA)\} = K+1+(M+2)L$. This happens if and only if P_1 keeps the weight of AB unchanged, while she removes a positive integer weight from at least one of BC and CA. Consequently, we can write $w_1(BC) = K+j_1+(M+2)L$ and $w_1(CA) = K+j_2+(M+2)L$, for some $1 \leqslant j_1 \leqslant i$ and $1 \leqslant j_2 \leqslant M+3-i$ with $j_1+j_2 \leqslant M+2$. Defining $m = j_1+j_2-2$, we thus have $m \leqslant M$. Moreover, the lower bounds of $j_1 \geqslant 1$ and $j_2 \geqslant 1$, along with $j_1+j_2 = (m+2)$, implies that each of j_1 and j_2 is bounded above by $(m+1)$. Let k be the unique positive integer such that

$$\frac{m(m+1)}{2} \leqslant k \leqslant \frac{m(m+3)}{2} \quad \text{and} \quad K+(M+2)L \equiv k \bmod (m+1) \tag{6.3}$$

Using the above-mentioned assertion that $m \leqslant M$, the former of the two criteria stated in (6.3), and (6.1), we conclude that

$$k \leqslant \frac{m\ \ m+3)}{2} \leqslant \frac{M\ \ M+3)}{2} < \frac{M+1)\ \ M+2)}{2} \leqslant K$$

The second criterion of (6.3) implies the existence of some $\ell \in \ \ _0$ such that $K + \ \ M+2)L = k + \ \ m+1)\ell$. In the second round, P_2 removes the strictly positive edge-weight $\ \ K \ \ \ k)$ from the edge DE, leaving P_1 with a configuration where $w_2\ \ DE) = k$ and

$$\begin{aligned} &w_2\ \ \ \ B) = K+1+\ \ M+2)L = k+1+\ \ m+1)\ell, \\ &w_2\ \ BC) = K+j_1+\ \ M+2)L = k+j_1+\ \ m+1)\ell, \\ &w_2\ \ C\ \ \ ) = K+j_2+\ \ M+2)L = k+j_2+\ \ m+1)\ell = k+m+2 \ \ \ \ j_1+\ \ m+1)\ell, \end{aligned}$$

which satisfies the hypothesis of (1) (with (4.1) ensured by the first condition stated in (6.3)). s $m \leqslant M$, hence P_1 loses by our induction hypothesis in (I1).

(ii) The second scenario, which needs to be considered only when L is a *positive* integer, is as follows: suppose P_1 removes some weight from at most two of the edges $\ \ B$, BC and $C\ \ $ in the first round, such that

$$K+1+\ \ M+2)L > \min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\} > K \tag{6.4}$$

Let r be the unique element in $\{1, 2, \ \ , M+2\}$ such that $\min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\} \ \ \ K \equiv r \bmod \ \ M+2)$, which implies the existence of some $\ell \in \ \ _0$ such that $\min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\} = K+r+\ \ M+2)\ell$. Note, using the first inequality of (6.4), that

$$K+1+\ \ M+2)L > K+r+\ \ M+2)\ell \quad \text{and} \quad r \geqslant 1 \implies \ell \leqslant \ \ L \ \ \ \ 1) \tag{6.5}$$

This, in turn, shows that

$$\begin{aligned} K+M+3 \ \ \ \ r+\ \ M+2)\ell &\leqslant K+M+3 \ \ \ \ r+\ \ M+2)\ \ L \ \ \ \ 1) = K+1 \ \ \ \ r+\ \ M+2)L \\ &< K+1+\ \ M+2)L = \min\{w_0\ \ \ \ B), w_0\ \ BC), w_0\ \ C\ \ \ )\} \end{aligned} \tag{6.6}$$

For ease of exposition, let us denote by e_1 the edge out of $\ \ B$, BC and $C\ \ $ whose edge-weight equals $\min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\}$ – it is evident from the first inequality of (6.4) (and from the observation that $\min\{w_0\ \ \ \ B), w_0\ \ BC), w_0\ \ C\ \ \ )\} = K+1+\ \ M+2)L$) that $w_1\ \ e_1) < w_0\ \ e_1)$. Let us denote by e_2 and e_3 the remaining two edges out of $\ \ B$, BC and $C\ \ $. Since P_1 could have modified the edge-weights of at most two edges out of $\ \ B$, BC and $C\ \ $ in the first round, we may assume, without loss of generality, that $w_0\ \ e_3) = w_1\ \ e_3)$. Note that, since $r \geqslant 1$, we have

$$K+1+\ \ M+2)\ell \leqslant K+r+\ \ M+2)\ell = \min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\} \leqslant w_1\ \ e_2),$$

and from (6.6), we know that $K+M+3 \ \ \ \ r+\ \ M+2)\ell < w_0\ \ e_3) = w_1\ \ e_3)$. Hence, in the second round, P_2 selects the vertex that is in common between e_2 and e_3, removes weight $w_1\ \ e_2) \ \ \ \ \{K+1+\ \ M+2)\ell\}$ from the edge e_2, and removes weight $w_1\ \ e_3) \ \ \ \ \{K+M+3 \ \ \ \ r+\ \ M+2)\ell\}$ from the edge e_3, leaving P_1 with a configuration where $w_2\ \ DE) = K$, and

$$w_2\ \ e_1) = K+r+\ \ M+2)\ell,\ w_2\ \ e_2) = K+1+\ \ M+2)\ell,\ w_2\ \ e_3) = K+M+3 \ \ \ \ r+\ \ M+2)\ell$$

This configuration satisfies the hypothesis of (1). s $\ell \leqslant \ \ L \ \ \ \ 1)$ (as justified in (6.5)), hence P_1 loses by our induction hypothesis in (I3) (once again, we emphasize that this entire case need not be considered at all when $L = 0$).

(iii) Suppose P_1 removes some weight from at most two of the edges $\ \ B$, BC and $C\ \ $ in the first round, such that $\min\{w_1\ \ \ \ B), w_1\ \ BC), w_1\ \ C\ \ \ )\} \leqslant K$. Let, once again, e_1 denote the edge, out of $\ \ B$, BC and $C\ \ $, whose edge-weight attains this minimum, and let e_2 and e_3 be the remaining two edges, with $w_1\ \ e_3) = w_0\ \ e_3)$ since P_1 cannot alter the edge-weights of all three of these edges in a single round. In the second round, P_2 selects the vertex that is in common between e_2 and e_3, removes from e_3 the weight $w_1(e_3) - \{K - w_1(e_1)\}$ (note that $w_1(e_3) = w_0(e_3) \geqslant K+1$), and removes the entire edge e_2. This leaves P_1 with a galaxy graph, with one component consisting of the edges e_1 and e_3, the other comprising the edge DE, and $w_2(e_1) + w_2(e_3) = K = w_2(DE)$. Consequently, P_1 loses by Theorem 2.3.

(iv) The final possibility is where P_1 removes a positive integer weight from DE in the first round, so that $w_1(DE) = k < K$. If $k = 0$, then P_2 is left with the edges AB, BC and CA of a triangle, with their edge-weights *not* all equal, and she wins by Theorem 5.1. Therefore, in what follows, it suffices to assume that $w_1(DE) = k$ with $k > 0$.

As mentioned right after (6.2), we assume, without loss of generality, that $w_0(AB) \leqslant w_0(BC) \leqslant w_0(CA)$, so that $2i \leqslant (M+3)$. Now, let m be the unique positive integer such that $m(m+1) \leqslant 2k \leqslant m(m+3)$, and let r be the unique element in $\{0,1,\ldots,m\}$ such that $K + (M+2)L - k \equiv r \bmod (m+1)$. Thus, there exists some $\ell \in \mathbb{N}_0$ such that $K + (M+2)L = k + r + (m+1)\ell$, so that

$$\begin{aligned}
w_1(AB) &= w_0(AB) = K+1+(M+2)L = k+r+1+(m+1)\ell,\\
w_1(BC) &= w_0(BC) = K+i+(M+2)L = k+r+i+(m+1)\ell,\\
w_1(CA) &= w_0(CA) = K+M+3-i+(M+2)L = k+M+3-i+r+(m+1)\ell
\end{aligned}$$

There are two subcases to consider. The first is where $2(r+1) > m+i-M$, in which case P_2 removes weight $(r+i-1)$ from the edge BC and weight $\{2(1+r)+M-i-m\}$ (which is strictly positive) from the edge CA in the second round, leaving P_1 with

$$\begin{aligned}
&w_2(AB) = k+(r+1)+(m+1)\ell,\ w_2(BC) = k+1+(m+1)\ell,\\
&w_2(CA) = k+m+2-(r+1)+(m+1)\ell,\ w_2(DE) = k
\end{aligned}$$

This configuration satisfies the hypothesis of (1), and since $k < K$, P_1 loses by our induction hypothesis in (I2) when $(M+1)(M+2)/2 \leqslant k$, and by our induction hypothesis in (I1) when $k \leqslant M(M+3)/2$ (which, in turn, implies that $m \leqslant M$). Once again, we recall for the reader one of the remarks made right before (6.2): when $K = (M+1)(M+2)/2$, we have $k < K \implies k \leqslant M(M+3)/2$, which in turn implies that k satisfies the second criterion of (4.1) for some $m \leqslant M$, and hence, our induction hypothesis in (I1) becomes applicable.

The second subcase is where

$$2(r+1) \leqslant m+i-M, \tag{6.7}$$

which, along with the inequality $2i \leqslant M+3$ (as mentioned in the first sentence of the previous paragraph), yields:

$$m+i-M \geqslant 2(r+1) \geqslant 2 \implies m + \frac{M+3}{2} - M \geqslant 2 \implies m - \frac{M}{2} \geqslant \frac{1}{2} \implies m \geqslant \frac{M+1}{2} \tag{6.8}$$

Starting with (6.7) and applying the inequality $2i \leqslant M+3$, as well as the inequality deduced in (6.8), we obtain:

$$\begin{aligned}
(r+i) &< \frac{m+i-M}{2} + i - 1 = \frac{m-M}{2} + \frac{3i}{2} - 1\\
&\leqslant \frac{m-M}{2} + \frac{3(M+3)}{4} - 1 = \frac{m}{2} + \frac{M+1}{4} + 1 \leqslant m+1,
\end{aligned}$$

which is what we need for the remaining argument. In the second round, P_2 removes weight r from the edge AB and weight $(M-m+1+2r)$ (which is strictly positive) from the edge CA, leaving P_1 with

$$w_2(AB) = k+1+(m+1)\ell,\ w_2(BC) = k+(r+i)+(m+1)\ell,$$

$$w_2\ C\ \ ) = k+m+2\quad r+i)+\ m+1)\ell,\ w_2\ DE) = k$$

This configuration satisfies the hypothesis of (1), and since $k < K$, once again, P_1 loses by our induction hypothesis in (I2) when $M+1)\ M+2)/2 \leqslant k$, and by our induction hypothesis in (I1) when $k \leqslant M\ M+3)/2$ (as before, we need only apply the induction hypothesis of (I1) when $K = M+1)\ M+2)/2$).

This completes the proof of the fact that the configuration in (6.2) is losing. This also concludes the inductive proof of Step (D2) in the proof of Theorem 4.10.

6.3. **Proof of Step** (D3) **for proving Theorem 4.6.** We assume, without loss of generality, that $w_0\quad B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$. Letting $w_0\ DE) = k$ for some $k \in$, let m be the unique positive integer for which k satisfies the second criterion of (4.1). If $w_0\quad B)\quad k \equiv r_1 \bmod\ m+1)$, $w_0\ BC)\quad k \equiv r_2 \bmod\ m+1)$ and $w_0\ C\ \ )\quad k \equiv r_3 \bmod\ m+1)$, where $r_1, r_2, r_3 \in \{1, 2, \quad, m+1\}$, then for some ℓ_1, ℓ_2 and ℓ_3 in $_0$, we have

$$w_0\quad B) = k+r_1+\ m+1)\ell_1,\ w_0\ BC) = k+r_2+\ m+1)\ell_2,\ w_0\ C\ \ ) = k+r_3+\ m+1)\ell_3 \tag{6.9}$$

The first case to consider is where $\ell_1 = \ell_2 = \ell_3 = \ell$. In this case, under our assumption that $w_0\quad B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$, we must have $r_1 \leqslant r_2 \leqslant r_3$. If $r_1 = r_2 = r_3$, then we have $w_0\quad B) = w_0\ BC) = w_0\ C\ \ )$, in which case P_1 simply removes the edge DE in the first round and wins by Theorem 5.1.

If r_2 and r_3 are such that $m+2\quad r_2 \leqslant r_3$, then P_1 chooses the vertex , and removes from the edges B and C the weights $r_1\quad 1)$ and $\{r_3\quad m+2\quad r_2)\}$ respectively (note that the total weight removed is strictly positive since our initial configuration does *not* satisfy the hypothesis of (1)). This leaves P_2 with $w_1\ DE) = k$ and

$$w_1\quad B) = k+1+\ m+1)\ell,\ w_1\ BC) = k+r_2+\ m+1)\ell,\ w_2\ C\ \ ) = k+m+2\quad r_2+\ m+1)\ell,$$

which now satisfies the hypothesis of (1), and P_2 loses by Step (D2) proved earlier.

Suppose, now, that $m+2\quad r_2 > r_3$, so that, if we set $m' = r_2 + r_3\quad 2r_1$, we have, using $r_1 \geqslant 1$,

$$m+2\quad r_2 > r_3 \implies r_2 + r_3 < m+2 \implies m' = r_2 + r_3\quad 2r_1 < m+2\quad 2r_1 \leqslant m \tag{6.10}$$

Let k' be the unique positive integer satisfying

$$\frac{m'\ m'+1)}{2} \leqslant k' \leqslant \frac{m'\ m'+3)}{2} \quad \text{and} \quad k+r_1\quad 1+\ m+1)\ell \equiv k' \bmod\ m'+1) \tag{6.11}$$

The first of the two criteria stated in (6.11), along with the inequality $m' < m$ that we have deduced in (6.10), and the fact that k and m satisfy the second criterion of (4.1), yields $k' < k$, which, in turn, combined with the second of the two criteria stated in (6.11), implies the existence of some $\ell' \in\ _0$ such that $k+r_1+\ m+1)\ell = k'+1+\ m'+1)\ell'$. Note that this yields, using the definition of m':

$$w_0\quad B) = k+r_1+\ m+1)\ell = k'+1+\ m'+1)\ell',$$

$$w_0\ BC) = k+r_2+\ m+1)\ell = k'+\ 1+r_2\quad r_1)+\ m'+1)\ell',$$

$$w_0\ C\ \ ) = k+r_3+\ m+1)\ell = k'+1+r_3\quad r_1+\ m'+1)\ell' = k'+m'+2\quad 1+r_2\quad r_1)+\ m'+1)\ell'$$

Therefore, P_1 simply removes weight $k\quad k')$ (which is strictly positive, as argued above) from the edge DE in the first round, leaving P_2 with a configuration that satisfies the hypothesis of (1), and hence, P_2 loses by Step (D2) proved earlier.

Now, let us consider the second case, i.e. where ℓ_1, ℓ_2 and ℓ_3 are *not* all equal. This, along with our assumption that $w_0\quad B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$, implies that $\ell_1 < \ell_3$. This, in turn, implies (using the inequalities $r_1 + r_3 \geqslant 2$ and $\ell_3\quad \ell_1 \geqslant 1$) that

$$\{k+r_3+\ m+1)\ell_3\}\quad \{k+m+2\quad r_1+\ m+1)\ell_1\} = r_3+r_1\quad m+2)+\ m+1)\ \ell_3\quad \ell_1) \geqslant 1$$

P_1, therefore, removes weight $\{k+r_3+\ m+1)\ell_3\}\quad\{k+m+2\quad r_1+\ m+1)\ell_1\}$ from the edge $C\ \ $, and weight $r_2\quad 1+\ m+1)\ \ell_2\quad \ell_1)$ from the edge BC, in the first round, leaving P_2 with

$$w_1\quad B)=k+r_1+\ m+1)\ell_1,\ w_1\ BC)=k+1+\ m+1)\ell_1,$$
$$w_1\ C\quad)=k+m+2\quad r_1+\ m+1)\ell_1,\ w_1\ DE)=k,$$

which satisfies the hypothesis of (1), and hence, P_2 loses by Step (D2) proved earlier. This completes the proof of Step (D3).

Before we begin the proof of Step (D4), we state and prove a crucial lemma:

Lemma 6.1. *Given $w_0\quad B)\leqslant w_0\ BC)\leqslant w_0\ C\quad)$, with $w_0\quad B)$, $w_0\ BC)$ and $w_0\ C\quad)$ not all equal, the multiset $\{w_0\quad B),w_0\ BC),w_0\ C\quad)\}$ is special (see Definition 4.5) if and only if we have*

$$2w_0\quad B)>\ w_0\ BC)+w_0\ C\quad)\quad 2w_0\quad B))\ w_0\ BC)+w_0\ C\quad)\quad 2w_0\quad B)+1)\tag{6.12}$$

Proof. When (6.12) holds, we set $m=w_0\ BC)+w_0\ C\quad)\quad 2w_0\quad B)$, and we let k be the unique positive integer that, along with this choice of m, satisfies the second criterion of (4.1) as well as the relation $w_0\quad B)\quad 1\equiv k \bmod\ m+1)$. Note that the inequality in (6.12) ensures that

$$w_0\quad B)>\frac{m\ m+1)}{2}\implies w_0\quad B)\geqslant\frac{m\ m+1)}{2}+1\implies w_0\quad B)\quad 1\geqslant\frac{m\ m+1)}{2},$$

so that, from the way we have defined k above, we know that there exists $\ell\in\ _0$ such that $w_0\quad B)=k+1+\ m+1)\ell$. Next, from our definition of m, we have

$$w_0\ BC)+w_0\ C\quad)=2w_0\quad B)+m=2\{k+1+\ m+1)\ell\}+m=2k+m+2+2\ m+1)\ell\tag{6.13}$$

Let $w_0\ BC)\quad k\equiv i \bmod\ m+1)$ for some $i\in\{1,2,\quad,m+1\}$. Note that, since $w_0\ BC)\geqslant w_0\quad B)=k+1+\ m+1)\ell$, we can find $\ell_1\in\ _0$ such that $w_0\ BC)=k+i+\ m+1)\ell_1$. Likewise, setting $w_0\ C\quad)\quad k\equiv j \bmod\ m+1)$ for some $j\in\{1,2,\quad,m+1\}$, we see that there exists some $\ell_2\in\ _0$ such that $w_0\ C\quad)=k+j+\ m+1)\ell_2$. From (6.13), we then obtain

$$\{k+i+\ m+1)\ell_1\}+\{k+j+\ m+1)\ell_2\}=2k+m+2+2\ m+1)\ell$$
$$\iff i+j+\ m+1)\ \ell_1+\ell_2)=m+2+2\ m+1)\ell$$

The first conclusion to draw from this is that $\ i+j)\equiv 1 \bmod\ m+1)$, and since $2\leqslant\ i+j)\leqslant 2\ m+1)$, the only possibility we are left with is that $\ i+j)=\ m+2)$. This, in turn, yields $\ \ell_1+\ell_2)=2\ell$. If possible, let us assume that $\ell_1\neq\ell_2$, which implies that $\ell_1<\ell_2$ (since $w_0\ BC)\leqslant w_0\ C\quad)$). This implies that $\ell_1\leqslant\ \ell\quad 1)$, which yields

$$w_0\ BC)=k+i+\ m+1)\ell_1\leqslant k+i+\ m+1)\ \ell\quad 1)=k+i\quad\ m+1)+\ m+1)\ell<k+1+\ m+1)\ell=w_0\quad B),$$

leading to a contradiction. Therefore, our assumption that $\ell_1\neq\ell_2$ must be erroneous, leading to $\ell_1=\ell_2=\ell$. We have, therefore, deduced the existence of $k,m,i\in\quad$ and $\ell\in\ _0$ such that (4.1) holds and $w_0\quad B)=k+1+\ m+1)\ell$, $w_0\ BC)=k+i+\ m+1)\ell$ and $w_0\ C\quad)=k+j+\ m+1)\ell=k+m+2\quad i+\ m+1)\ell$. This proves one side of the implication stated in Lemma 6.1. The proof that the converse is also true is much more straightforward and hence omitted from this paper.

6.4. **Proof of Step** (D4) **for proving Theorem 4.6.** The proof of Step (D4) happens via induction on the value of $w_0\ DE)$. The base case for this inductive argument has been addressed in §8.3. Suppose, for some $K\in\quad$ with $K\geqslant 4$, we have shown that every initial weight configuration on G_4 that satisfies *all* of the criteria mentioned in (2), along with $w_0\ DE)\leqslant K$, is losing. We now consider an initial weight configuration $\ w_0\quad B),w_0\ BC),w_0\ C\quad),w_0\ DE))$, with $w_0\ DE)=\ K+1)$, that satisfies all of the criteria stated in (2). The first round of the game played on this initial configuration can unfold in one of the following ways:

(i) Suppose P_1 removes some positive integer weight from at most two of the edges out of B, BC and C . If the resulting multiset $\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\}$ is special, thus guaranteeing the existence of k, m, i and ℓ as in Definition 4.5, then $w_1\ \ B) + w_1\ BC) + w_1\ C\ \ ) < w_0\ DE) = \ K+1)$ implies that $k < \ K+1)$. Thus, P_2 is allowed to remove weight $\ K+1)\ \ k$ from the edge DE in the second round, which she does, leaving P_1 with a configuration that satisfies the hypothesis of (1), so that P_1 loses by Step (D2).

Suppose the multiset $\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\}$ is not special. If $w_1\ \ B) = w_1\ BC) = w_1\ C\ \ )$, then P_2 removes the edge DE in the second round, defeating P_1 by Theorem 5.1. If not all of $w_1\ \ B)$, $w_1\ BC)$ and $w_1\ C\ \ )$ are equal, then P_2 removes from DE the weight $\ K+1)\ \ \ \{w_1\ \ B) + w_1\ BC) + w_1\ C\ \ )\}$ in the second round, leaving P_1 with a configuration $\ w_2\ \ B), w_2\ BC), w_2\ C\ \ ), w_2\ DE))$, with $w_2\ DE) \leqslant K$, that satisfies all of the criteria stated in (2). Consequently, P_1 loses by our induction hypothesis.

(ii) Suppose P_1 removes a positive integer weight from DE in the first round. If $w_1\ DE) = 0$, then P_2 wins by Theorem 5.1. So, we henceforth assume that $w_1\ DE) > 0$.

Without loss of generality, for the rest of this analysis, let us assume that $w_0\ \ B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$. Note that if $w_0\ \ B) \leqslant w_1\ DE) \leqslant w_0\ BC) + w_0\ C\ \ )$, then P_1 loses by Step (D1).

When $w_1\ DE) < w_0\ \ B)$, we have $w_1\ DE) < \min\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$. Since our initial weight configuration satisfies the criteria stated in (2) (in particular, the multiset $\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ is *not* special), hence the weight configuration $\ w_1\ \ B), w_1\ BC), w_1\ C\ \ ), w_1\ DE))$, obtained after the first round of the game, does *not* satisfy the hypothesis of (1). Therefore, all the criteria stated in Step (D3) are satisfied by $\ w_1\ \ B), w_1\ BC), w_1\ C\ \ ), w_1\ DE))$, and hence, P_2 wins.

Finally, we consider $w_1\ DE) > w_0\ BC) + w_0\ C\ \ )$. In this case, P_2 removes weight $w_0\ DE)\ \ w_1\ DE)$ from the edge B in the second round, so that $w_2\ \ B) = w_0\ \ B)\ \ \ w_0\ DE) + w_1\ DE) = w_1\ DE)\ \ \ w_0\ BC)\ \ \ w_0\ C\ \ )$, which implies that $w_2\ \ B) + w_2\ BC) + w_2\ C\ \ ) = w_2\ DE)$. Since, to begin with, we had $w_0\ \ B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$, we certainly have $w_2\ \ B) < w_2\ BC) = w_0\ BC) \leqslant w_2\ C\ \ ) = w_0\ C\ \ )$ – i.e. $w_2\ \ B)$, $w_2\ BC)$ and $w_2\ C\ \ )$ are *not* all equal. By Lemma 6.1, the multiset $\{w_2\ \ B), w_2\ BC), w_2\ C\ \ )\}$ is special iff

$$\begin{aligned} 2w_2\ \ B) &> \{w_2\ BC) + w_2\ C\ \ )\ \ \ 2w_2\ \ B)\}\{w_2\ BC) + w_2\ C\ \ )\ \ \ 2w_2\ \ B) + 1\} \\ &= \{w_0\ BC) + w_0\ C\ \ )\ \ \ 2w_2\ \ B)\}\{w_0\ BC) + w_0\ C\ \ )\ \ \ 2w_2\ \ B) + 1\} \end{aligned} \tag{6.14}$$

Considering, for any real $y > 0$, the function $f_y\ x) = \ y\ \ x)\ y\ \ x+1)\ \ x$ for $x \in [0, y]$, we see that

$$f_y'\ x) = \ \ 2y + 2x\ \ 2 = \ \ 2\ y + 1\ \ x),$$

which is strictly negative for as long as we have $y \geqslant x$. Therefore, $f_y\ x)$ is strictly decreasing over $x \in [0, y]$, which implies that if for some $x \in [0, y)$, we have $x > \ y\ \ x)\ y\ \ x+1)$, then for any $x' \in \ x, y]$, we would have $x' > \ y\ \ x')\ y\ \ x'+1)$. Setting $y = w_0\ BC) + w_0\ C\ \ )$, $x = 2w_2\ \ B)$ and $x' = 2w_0\ \ B)$ and applying this observation, we find that (6.14) holds if and only if (6.12) holds, and if (6.12) holds, then, by Lemma 6.1, we conclude that the multiset $\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ must be special, contradicting the assumption that our initial weight configuration satisfies all constraints of (2). Therefore, the multiset $\{w_2\ \ B), w_2\ BC), w_2\ C\ \ )\}$ *cannot* be special, and the configuration $\ w_2\ \ B), w_2\ BC), w_2\ C\ \ ), w_2\ DE))$ satisfies all the criteria stated in (2), along with $w_2\ DE) \leqslant K$. Hence, P_1 loses by our induction hypothesis.

This concludes the inductive argument, and thereby, the proof of Step (D4).

6.5. **Proof of Step** (D5) **for proving Theorem 4.6.** We assume, without loss of generality, that $w_0\ \ B) \leqslant w_0\ BC) \leqslant w_0\ C\ \ )$, so that the first criterion stated in Step (D5) boils down to $w_0\ DE) > w_0\ BC) + w_0\ C\ \ )$.

If $w_0\ \ B) = w_0\ BC) = w_0\ C\ \ )$, then P_1 simply removes DE in the first round, and P_2 loses by Theorem 5.1. Henceforth, therefore, we assume that $w_0\ \ B)$, $w_0\ BC)$ and $w_0\ C\ \ )$ are not all equal.

If $w_0\ DE) > w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )$, then P_1 removes weight $w_0\ DE)\ \ \ \{w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )\}$ from the edge DE in the first round, so that $w_1\ DE) = w_1\ \ B) + w_1\ BC) + w_1\ C\ \ )$. Since the edge-weights on $\ B$, BC and $C\ $ remain unchanged, and since $\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ is not special to begin with, neither is $\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\}$. Thus, P_2 is left with a configuration that satisfies all the constraints stated in (2), and hence, by Theorem D4, P_2 loses.

Let us now consider $w_0\ BC)+w_0\ C\ \ ) < w_0\ DE) < w_0\ \ B)+w_0\ BC)+w_0\ C\ \ )$. In this case, P_1 removes weight $w_0\ \ B) + w_0\ BC) + w_0\ C\ \ )\ \ \ w_0\ DE)$ from the edge $\ B$ in the first round, so that P_2 is left with $w_1\ \ B) + w_1\ BC) + w_1\ C\ \ ) = w_1\ DE)$. Evidently, $w_1\ \ B) < w_0\ \ B) \leqslant w_0\ BC) = w_1\ BC) \leqslant w_0\ C\ \ ) = w_1\ C\ \ )$, so that $w_1\ \ B)$, $w_1\ BC)$ and $w_1\ C\ \ )$ are not all equal. Next, we note that, by Lemma 6.1, the multiset $\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\}$ would be special if and only if we have

$$\begin{aligned} 2w_1\ \ B) &> \{w_1\ BC) + w_1\ C\ \ )\ \ \ 2w_1\ \ B)\}\{w_1\ BC) + w_1\ C\ \ )\ \ \ 2w_1\ \ B) + 1\} \\ &= \{w_0\ BC) + w_0\ C\ \ )\ \ \ 2w_1\ \ B)\}\{w_0\ BC) + w_0\ C\ \ )\ \ \ 2w_1\ \ B) + 1\}, \end{aligned} \tag{6.15}$$

and, much as we argued towards the end of the proof of Step (D4), we conclude that (6.15) holds if and only if (6.12) holds, which, in turn, applying Lemma 6.1, is true if and only if the multiset $\{w_0\ \ B), w_0\ BC), w_0\ C\ \ )\}$ is special. However, this contradicts one of the criteria mentioned in the statement of Step (D5), which allows us to conclude that, in fact, the multiset $\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\}$ cannot be special. Combining all these observations, we see that the configuration $\ w_1\ \ B), w_1\ BC), w_1\ C\ \ ), w_1\ DE))$ satisfies all of the criteria mentioned in (2), and hence, by Step (D4) that we have already proved, P_2 loses. This completes the proof of Step (D5).

7. Proof of our results reg rding the gr ph H_1

This section is dedicated to the proofs of all of our results in §4.2. t the very outset of §7, we inform the reader that not all parts of the proof of Theorem 4.10 have been included in this section. In §7, we have included

(i) the proof of our claim that any configuration on H_1 that is of the form given by (4.4), with $r = 0$, is losing,
(ii) and the proofs of our claims that any configuration on H_1 that is of one of the forms given by (4.5), (4.6) and (4.7), with $r = 2$, is losing.

The claims that configurations on H_1 of the form given by (4.4), with $r = 1$ or $r = 3$, are losing, have been established in §8.5 (when $r = 1$) and §8.8 (when $r = 3$) of the ppendix, §8. Likewise, the claims that configurations on H_1 of the form given by one of (4.5), (4.6) and (4.7), with $r = 4$, are losing, have been proved in §8.9 (when $s = 1$), §8.10 (when $s = 2$), §8.11 (when $s = 3$) and §8.12 (when $s \geqslant 4$) of §8.

7.1. **Proof of Lemma 4.7.** Since the edges $\ B$ and CD play symmetric roles in the graph H_1, it suffices for us to prove that an initial weight configuration on H_1 is winning whenever it satisfies the inequalities in (4.2), and the proof would be analogous if, instead, it satisfies the inequalities in (4.3). ssuming that (4.2) holds, P_1 chooses the vertex C, removes weight $w_0\ \ B) + w_0\ BC)\ \ \ w_0\ EF)$ from the edge BC, and removes the entire edge CD, in the first round, leaving P_2 with a galaxy graph comprising two components: one of these is the edge EF, the other is a star graph consisting of the edges $\ B$ and BC, and $w_1\ \ B) + w_1\ BC) = w_0\ EF) = w_1\ EF)$. Hence, P_2 loses by Theorem 5.1.

7.2. **Proof of Theorem 4.8.** If the set S defined in the statement of Theorem 4.8 is empty, then $a_i + b_i + c_i$ is even for each $i \in \{0, 1,\ \ , s\}$. In this case, P_1 removes the edge BC in the first round, leaving P_2

with a galaxy graph comprising three components, namely, the edges B, CD and EF, such that the triple $w_0 \quad B), w_0\ CD), w_0\ EF))$ is balanced, and P_2 loses by Theorem 2.3.

Suppose, now, that S is non-empty, so that the index I is well-defined, and we assume, without loss of generality, that $b_I = 1$. Let us choose $e_i \in \{0,1\}$, for $i \in \{0,1, \quad ,s\}$, such that $e_i + a_i + c_i$ is even – note, immediately, that $e_i = b_i$ for all $i \in \{I+1, \quad ,s\}$ (if this set is non-empty), and $e_I = 0$ (since I itself is an element of S). Consequently,

$$\sum_{i=0}^{s} e_i 2^i \leqslant \sum_{i=I+1}^{s} b_i 2^i + \sum_{i=0}^{I \quad 1} 2^i = \sum_{i=I+1}^{s} b_i 2^i + 2^I \quad 1 < \sum_{i=0}^{s} b_i 2^i = w_0 \quad B),$$

so that if P_1 chooses the vertex B, removes weight $w_0 \quad B) \quad {}_{i=0}^{s} e_i 2^i$ from the edge B, and removes the entire edge BC, in the first round, it leaves P_2 with a galaxy graph comprising three components, namely, the edges B, CD and EF, such that the triple $w_0 \quad B), w_0\ CD), w_0\ EF))$ is balanced (by our choice of the e_is). Consequently, P_2 loses by Theorem 2.3. This completes the proof of Theorem 4.8.

7.3. **Proof of Lemma 4.9.** Consider an initial weight configuration on H_1 that is of the form described in the statement of Lemma 4.9, with $m_1 \neq m_2$, and we assume, without loss of generality, that $m_1 > m_2$. Writing $f\ k) = r$, where $f\ k)$ is as defined just before the statement of Proposition 4.9, let us write $k = a_r a_{r\ 1} \quad a_1 a_0)_2$ (so that $a_r = 1$), $\ell_1 = b_r b_{r\ 1} \quad b_1 b_0)_2$ and $\ell_2 = c_r c_{r\ 1} \quad c_1 c_0)_2$. We further write $m_1 = {}_{s\ s\ 1} \quad {}_{1\ 0})_2$ and $m_2 = \beta_s \beta_{s\ 1} \quad \beta_1 \beta_0)_2$, such that at least one of ${}_s$ and β_s equals 1, and as $m_1 > m_2$, there exists $J \in \{0,1, \quad ,s\}$ such that ${}_i = \beta_i$ for all $i \in \{J+1, \quad ,s\}$ (if this set is non-empty), ${}_J = 1$ and $\beta_J = 0$. rmed with these, we see that

$$\begin{aligned} w_0 \quad B) &= {}_{s\ s\ 1} \quad {}_{1\ 0} b_r b_{r\ 1} \quad b_1 b_0)_2, \\ w_0\ CD) &= \beta_s \beta_{s\ 1} \quad \beta_1 \beta_0 c_r c_{r\ 1} \quad c_1 c_0)_2, \\ w_0\ EF) &= \underbrace{0\,0\,0 \quad 0\,0\,0}_{s+1)\text{ zeros}} a_r a_{r\ 1} \quad a_1 a_0)_2 \end{aligned}$$

Consequently, the largest index I for which the sum of the I-th coordinates of the base-2 representations of $w_0 \quad B)$, $w_0\ CD)$ and $w_0\ EF)$ is odd is given by $I = r+1+J$, and we have, as mentioned above, ${}_J = 1$, which implies that the I-th coordinate in the base-2 representation of $w_0 \quad B)$ equals 1. Thus, P_1 wins by Theorem 4.8.

Now, let us consider an initial weight configuration on H_1 that is of the form described in Lemma 4.9, with $m_1 = m_2 = m$ and $\min\{\ell_1, \ell_2\} \geqslant k$. Let us, as above, set $r = f\ k)$, and let $k = a_r a_{r\ 1} \quad a_1 a_0)_2$ (so that, by definition of $f\ k)$, we must have $a_r = 1$), $\ell_1 = b_r b_{r\ 1} \quad b_1 b_0)_2$ and $\ell_2 = c_r c_{r\ 1} \quad c_1 c_0)_2$ (keeping in mind that $0 \leqslant \ell_1, \ell_2 \leqslant 2^{r+1} \quad 1$). Since $\min\{\ell_1, \ell_2\} \geqslant k$, we must have $b_r = c_r = 1$ as well. Let us also set $m = {}_{s\ s\ 1} \quad {}_{1\ 0})_2$, with ${}_s = 1$. We then have

$$\begin{aligned} w_0 \quad B) &= {}_{s\ s\ 1} \quad {}_{1\ 0} b_r b_{r\ 1} \quad b_1 b_0)_2, \\ w_0\ CD) &= {}_{s\ s\ 1} \quad {}_{1\ 0} c_r c_{r\ 1} \quad c_1 c_0)_2, \\ w_0\ EF) &= \underbrace{0\,0\,0 \quad 0\,0\,0}_{s+1)\text{ zeros}} a_r a_{r\ 1} \quad a_1 a_0)_2, \end{aligned}$$

so that the largest index I for which the sum of the I-th coordinates of the base-2 representations of $w_0 \quad B)$, $w_0\ CD)$ and $w_0\ EF)$ is odd is given by $I = r$ (since $a_r = b_r = c_r = 1$, as mentioned above). Since $b_r = c_r = 1$, P_1 wins by Theorem 4.8.

Finally, let us consider an initial weight configuration on H_1 that is of the form described in Lemma 4.9, with $m_1 = m_2 = m$, $k \in \{\ell_1, \ell_2\}$ and either $\min\{\ell_1, \ell_2\} > 0$ or $w_0\ BC) > 0$. Without loss of generality, let us assume that $k = \ell_1$. Then, in the first round, P_1 chooses the vertex C, removes weight ℓ_2 from CD, and removes the entire edge BC (our assumption ensures that the total weight removed is strictly positive), so that P_2 is left with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{f(k)+1}m + \ell_1$, $w_1(CD) = 2^{f(k)+1}m$ and $w_1(EF) = k = \ell_1$. As the triple $(2^{f(k)+1}m + \ell_1, 2^{f(k)+1}m, \ell_1)$ is balanced, P_2 loses by Theorem 2.3. This completes the proof of Lemma 4.9.

7.4. **Proof that a configuration of the form given by** (4.4)**, with** $r = 0$**, is losing.** We begin by proving that an initial weight configuration on H_1, that is of the form given by (4.4), is losing for $r = 0$ and $s = k$. First, we consider the configuration where $w_0(AB) = w_0(CD) = 2m$ and $w_0(BC) = w_0(EF) = 1$, for $m \in \mathbb{N}$:

(i) Suppose P_1 removes a positive integer weight from AB and a non-negative integer weight from BC during the first round (an analogous situation would arise if, instead, P_1 removes a positive integer weight from CD and a non-negative integer weight from BC). This yields $w_1(AB) = 2n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0, 1\}$. Note that the resulting configuration is of the form described in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 0$, and since $m_1 \neq m_2$, P_1 loses.

(ii) Suppose P_1 removes the edge EF in the first round. Then P_2 removes the edge BC in the second round, leaving P_1 with a galaxy graph comprising the edges AB and CD, where $w_2(AB) = w_2(CD) = 2m$, and P_1 loses by Theorem 2.3.

(iii) Suppose P_1 removes the edge BC in the first round, without disturbing the edge-weights of AB or CD. Then P_2 removes the edge EF in the second round, leaving P_1 with a galaxy graph comprising the edges AB and CD, where $w_2(AB) = w_2(CD) = 2m$, and P_1 loses by Theorem 2.3.

Suppose we have shown, for some $K \in \mathbb{N}$, that the configuration given by (4.4), with $r = 0$ and $s = k$, with $k \leqslant K$, is losing. We now consider the first round of the game played on the initial configuration on H_1 where $w_0(AB) = w_0(CD) = 2^{R+1}m$ and $w_0(BC) = w_0(EF) = (K+1)$, where $m \in \mathbb{N}$ and we set $R = f(K+1)$:

(i) Suppose P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, during the first round (as before, an analogous situation would be if P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC). Once again, this leads to $w_1(AB) = 2^{R+1}n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0, 1, \ldots, 2^{R+1} - 1\}$, so that the resulting configuration is of the form described in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 0$, and since $m_1 \neq m_2$, P_1 loses.

(ii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = k \leqslant K$. Then, P_2 removes weight $(K+1-k)$ from the edge BC in the second round, leaving P_1 with

$$w_2(AB) = w_2(CD) = 2^{R+1}m = 2^{f(k)+1}2^{R-f(k)}m,\ w_2(BC) = w_2(EF) = k,$$

where $k \leqslant K \implies f(k) \leqslant f(K+1) = R$. When $k \geqslant 1$, this is of the form given by (4.4) with $r = 0$ and $w_2(BC) = w_2(EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. When $k = 0$, P_1 loses by Theorem 2.3.

(iii) Suppose P_1 removes a positive integer weight from BC, without disturbing the edge-weights of AB and CD, during the first round. Then P_2 removes weight $(K+1-k)$ from EF in the second round, and, as in the previous case, P_1 loses by either our induction hypothesis (when $1 \leqslant w_1(BC) \leqslant K$), or by Theorem 2.3 (when $w_1(BC) = 0$).

This completes the inductive proof of our claim that any configuration on H_1 of the form given by (4.4) is losing whenever $r = 0$ and $s = k$ (equivalently, $r = 0$ and $w_0(AB) = w_0(CD)$).

Let us assume, now, that for some $K \in \mathbb{N}$, we have proved that any initial configuration of the form given by (4.4), with $r = 0$, is losing whenever $k \leqslant K$. We fix any $L \in \{0, 1, \ldots, K-1\}$, and consider the first round of the game played on the initial weight configuration on H_1 where $w_0(AB) = 2^{R+1}m$, $w_0(CD) = 2^{R+1}m + (L+1)$, $w_0(BC) = (K-L)$ and $w_0(EF) = (K+1)$, for any $m \in \mathbb{N}$, where we have set $R = f(K+1)$:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. We consider the following few subcases:

(a) Suppose $w_1\ CD) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1, \quad ,L\}$, and $w_1\ BC) = t \in \quad$ with $t \leqslant\ K \quad L)$. Note that $\ell \leqslant L$ and $t \leqslant\ K \quad L)$ together imply $\ \ell+t) \leqslant K$, which, in turn, implies $f\ t+\ell) \leqslant f\ K+1) = R$. In this case, P_2 removes weight $\ K+1) \quad t+\ell)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ t+\ell)+1}2^{R\ \ f\ t+\ell)}m,\ w_2\ BC) = t,$$
$$w_2\ CD) = 2^{f\ t+\ell)+1}2^{R\ \ f\ t+\ell)}m+\ell,\ w_2\ EF) =\ t+\ell),$$

which is of the form given by (4.4) with $r = 0$ and $w_2\ EF) =\ t+\ell) \leqslant K$, and hence, P_1 loses by our induction hypothesis.

(b) Suppose $w_1\ CD) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1, \quad ,L\}$, but this time, $w_1\ BC) = 0$. Here, P_2 removes weight $\ K+1 \quad \ell)$ from the edge EF in the second round, leaving P_1 with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_2 \quad B) = 2^{R+1}m$, $w_2\ CD) = 2^{R+1}m+\ell$ and $w_2\ EF) = \ell$, and P_1 loses by Theorem 2.3.

(c) Suppose $w_1\ CD) = 2^{R+1}n+\ell$ for some $n < m$ and $\ell \in \{0,1, \quad ,2^{R+1} \quad 1\}$. The resulting configuration is of the form described in Lemma 4.9, with $m_1 = n$, $m_2 = m$, $\ell_1 = \ell$ and $\ell_2 = 0$, and since $m_1 \neq m_2$, P_1 loses.

(ii) Suppose P_1 removes a positive integer weight from $\ B$, and a non-negative integer weight from BC, in the first round. Then we would have $w_1 \quad B) = 2^{R+1}n+\ell$ for some $n < m$ and $\ell \in \{0,1, \quad ,2^{R+1} \quad 1\}$, and once again, P_1 would lose by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$. If $k \in \{K \quad L, K \quad L+1, \quad ,K\}$, then P_2 removes weight $\ L+1) \quad \{k \quad K \quad L)\}$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ k)+1}2^{R\ \ f\ k)}m,\ w_2\ BC) =\ K \quad L),$$
$$w_2\ CD) = 2^{f\ k)+1}2^{R\ \ f\ k)}m+k \quad K \quad L),\ w_2\ EF) = k,$$

which is of the same form as (4.4) with $r = 0$ and $w_2\ EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. If $k \in \{1,2, \quad ,K \quad L \quad 1\}$, then P_2 removes weight $\ L+1)$ from the edge CD, and weight $\ K \quad L) \quad k$ from the edge BC, in the second round, leaving P_1 with

$$w_2 \quad B) = w_2\ CD) = 2^{f\ k)+1}2^{R\ \ f\ k)}m,\ w_2\ BC) = w_2\ EF) = k,$$

which is of the same form as (4.4) with $r = 0$ and $k \leqslant K$, so that P_1 loses by our induction hypothesis (we could have also argued that the configuration obtained above, after the second round, is of the form given by (4.4) with $r = 0$ and $w_2 \quad B) = w_2\ CD)$, and we have already proved above that such a configuration is losing). Finally, if $k = 0$, P_2 wins because of Remark 4.12.

(iv) Finally, suppose P_1 removes a positive integer weight from BC in the first round, but leaves the edge-weights of $\ B$ and CD undisturbed. If $w_1\ BC) = t \in \{1,2, \quad ,K \quad L \quad 1\}$, then P_2 removes weight $\ K \quad L \quad t)$ from EF in the second round, so that $w_2\ EF) = L+t+1 \leqslant L+\ K \quad L \quad 1)+1 = K$, which, in turn, implies that $f\ L+t+1) \leqslant f\ K+1) = R$. Thus, P_1 is left with

$$w_2 \quad B) = 2^{f\ L+t+1)+1}2^{R\ \ f\ L+t+1)}m,\ w_2\ BC) = t,$$
$$w_2\ CD) = 2^{f\ L+t+1)+1}2^{R\ \ f\ L+t+1)}m+\ L+1),\ w_2\ EF) = L+t+1,$$

which is of the same form as (4.4) with $r = 0$ and $w_2\ EF) =\ L+t+1) \leqslant K$, so that P_1 loses by our induction hypothesis. If $w_1\ BC) = 0$, then P_2 removes weight $K \quad L$ from the edge EF in the second round, leaving P_1 with a galaxy graph comprising the edges $\ B$, CD and EF, where $w_2 \quad B) = 2^{R+1}m$, $w_2\ CD) = 2^{R+1}m+\ L+1)$ and $w_2\ EF) =\ L+1)$, so that P_1 loses by Theorem 2.3.

This completes the proof of our claim that any configuration on H_1 of the form given by (4.4), with $r = 0$, is losing.

7.5. **Proof of Lemma 4.11.** Consider an initial weight configuration on H_1 that is of the form given by Lemma 4.11. To begin with, if $m_1 \neq m_2$, we know that P_1 wins by Lemma 4.9. Assuming, therefore, that $m_1 = m_2 = m$, as well as that $\ell_1 \leqslant \ell_2$, we obtain $w_0(AB) = 2^{f(k)+1}m + \ell_1$, $w_0(CD) = 2^{f(k)+1}m + \ell_2$, $w_0(EF) = k$ and $w_0(BC) > k - \min\{\ell_1, \ell_2\} = k - \ell_1$. In the first round, P_1 removes weight ℓ_2 from CD and weight $w_0(BC) - (k - \ell_1)$ from BC, so that P_2 is left with $w_1(AB) = 2^{f(k)+1}m + \ell_1$, $w_1(CD) = 2^{f(k)+1}m$, $w_1(EF) = k$ and $w_1(BC) = (k - \ell_1)$, which is of the form given by (4.4) with $r = 0$. Consequently, P_2 loses.

Consider, now, any initial weight configuration on H_1 that satisfies $w_0(BC) > w_0(EF)$. If $w_0(EF) \geqslant w_0(AB)$, then we have $w_0(AB) \leqslant w_0(EF) < w_0(AB) + w_0(BC)$, and if $w_0(EF) \geqslant w_0(CD)$, then we have $w_0(CD) \leqslant w_0(EF) < w_0(BC) + w_0(CD)$. In either of these cases, P_1 wins by Lemma 4.7. Therefore, throughout the rest of the proof, let us assume that $w_0(EF) < \min\{w_0(AB), w_0(CD)\}$.

Writing $w_0(EF) = k$ with $k \in \mathbb{N}$ (so that $w_0(BC) > k$), $w_0(AB) = 2^{f(k)+1}m_1 + \ell_1$ and $w_0(CD) = 2^{f(k)+1}m_2 + \ell_2$, where $m_1, m_2 \in \mathbb{N}_0$ and $\ell_1, \ell_2 \in \{0, 1, \ldots, 2^{f(k)+1} - 1\}$, we note the following:

(i) If $m_1 \neq m_2$, then P_1 wins by Lemma 4.9.

(ii) If $m_1 = m_2 = 0$, then $w_0(EF) < \min\{w_0(AB), w_0(CD)\}$ implies that $k < \min\{\ell_1, \ell_2\}$, and once again, P_1 wins by Lemma 4.9.

(iii) Finally, if $m_1 = m_2 = m \in \mathbb{N}$ and, assuming (without loss of generality) that $\ell_1 \leqslant \ell_2$, if $k > \ell_1$, then P_1 removes weight ℓ_2 from the edge CD, and weight $w_0(BC) - (k - \ell_1)$ from the edge BC, in the first round, leaving P_2 with

$$w_1(AB) = 2^{f(k)+1}m + \ell_1,\ w_1(BC) = k - \ell_1,\ w_1(CD) = 2^{f(k)+1}m,\ w_1(EF) = k,$$

which is of the same form as (4.4) with $r = 0$, and by what we have already proved in §7.4, we know that P_2 loses.

We recall for the reader here that the proof of Lemma 4.13 has been included in §8.4 of §8.

7.6. **Proof that a configuration that is either of the form given by** (4.5) **or of the form given by** (4.6)**, for** $r = 2$**, is losing.** Before we begin §7.6, we remind the reader that our claim, that any configuration of the form given by (4.4) with $r = 1$ is losing, has already been proved in §8.5 of §8. This is going to be of use to us in the proofs that follow. That configurations that are either of the form given by (4.5) or of the form given by (4.6), with $r = 2$, are losing, has been proved *together*, via a single inductive argument in §7.6. The base cases for this inductive argument have been addressed in §8.6 of §8.

Suppose, for some $K \in \mathbb{N}$, we have proved that

(i) any initial weight configuration on H_1, that is of the form given by (4.5) with $r = 2$, and having $w_0(EF) = k \leqslant K$, is losing,

(ii) any initial weight configuration on H_1, that is of the form given by (4.6) with $r = 2$, and having $w_0(EF) = k \leqslant K$, is losing.

The first scenario to consider is where $K \equiv 0 \bmod 4$, and the initial weight configuration is

$$w_0(AB) = 2^{R+1}m + 2,\ w_0(BC) = 1,\ w_0(CD) = 2^{R+1}m + K + 2,\ w_0(EF) = K + 1 \tag{7.1}$$

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. We consider a few subcases of this case:

   (a) Suppose $w_1(CD) = 2^{R+1}m + \ell$ for some $\ell \leqslant K + 1$ with $\ell \equiv 2 \bmod 4$ (which actually tells us that we must have $\ell \leqslant K - 2$), and $w_1(BC) = 1$. Here, P_2 removes weight $(K + 1) - (\ell - 1)$ from the edge EF in the second round, so that P_1 is left with

$$w_2(AB) = 2^{f(\ell-1)+1}2^{R-f(\ell-1)}m + 2,\ w_2(BC) = 1,$$

$$w_2\ CD) = 2^{f\ \ell\ \ 1)+1} 2^{R\ \ f\ \ell\ \ 1)} m + \ell,\ w_2\ EF) = \ \ \ell\ \ \ \ 1)$$

Since $\ell\ \ \ 1) \equiv 1 \bmod 4$, this configuration is of the form given by (4.5) with $r = 2$ and $w_2\ EF) = \ \ell\ \ \ 1) \leqslant\ \ K\ \ \ 3) < K$, so that P_1 loses by our induction hypothesis.

(b) Suppose $w_1\ CD) = 2^{R+1}m + \ell$ for some $\ell \leqslant K\ \ \ 3$ with $\ell \equiv i \bmod 4$ for some $i \in \{0,1,3\}$, and $w_1\ BC) = 1$. Here, P_2 removes weight $\ K+1)\ \ \ \ \ \ell+3)$ from the edge EF in the second round, so that P_1 is left with

$$w_2\ \ \ B) = 2^{f\ \ell+3)+1} 2^{R\ \ f\ \ell+3)} m + 2,\ w_2\ BC) = 1,$$

$$w_2\ CD) = 2^{f\ \ell+3)+1} 2^{R\ \ f\ \ell+3)} m + \ell,\ w_2\ EF) = \ \ \ell+3)$$

Since $\ \ell+3) \equiv j \bmod 4$ for some $j \in \{0,2,3\}$, this configuration is of the form given by (4.6) with $r = 2$ and $w_2\ EF) = \ \ \ell+3) \leqslant K$, so that P_1 loses by our induction hypothesis.

(c) Suppose $w_1\ CD) = 2^{R+1}m + \ell$ for $\ell \in \{K\ \ \ 1, K\}$, and $w_1\ BC) = 1$. Then P_2 removes weight $2\ \ \ \ \ K\ \ \ \ell)$ from the edge $\ \ B$ in the second round, leaving P_1 with

$$w_2\ \ \ B) = 2^{R+1}m +\ \ K\ \ \ \ell),\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m + \ell,\ w_2\ EF) = \ \ K+1),$$

which is of the form given by (4.4) with $r = 1$ when $\ell = K\ \ \ 1$, and of the form given by (4.4) with $r = 0$ when $\ell = K$, and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(d) Suppose $w_1\ CD) = 2^{R+1}m + \ell$ for some $\ell \leqslant K$, and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges $\ \ B, CD$ and EF, where $w_1\ \ \ B) = 2^{R+1}m + 2$, $w_1\ CD) = 2^{R+1}m + \ell$ and $w_1\ EF) = \ \ K+1)$. If $\ell \leqslant K\ \ \ 2$, P_2 wins by Lemma 4.13.

Suppose, now, that $\ell = K$. Since $K \equiv 0 \bmod 4$, the triple $\ 2^{R+1}m+1, 2^{R+1}m+K, K+1)$ is balanced, which means that the triple $\ 2^{R+1}m+2, 2^{R+1}m+K, K+1)$ is unbalanced, so that P_2 wins by Theorem 2.3. Likewise, if $\ell = \ \ K\ \ \ 1)$, since $K \equiv 0 \bmod 4$, the triple $\ 2^{R+1}m + 2, 2^{R+1}m+K\ \ \ 1, K\ \ \ 3)$ is balanced, which means that the triple $\ 2^{R+1}m+2, 2^{R+1}m+K\ \ \ 1, K+1)$ is unbalanced, and P_2 wins by Theorem 2.3.

(e) If $w_1\ CD) = 2^{R+1}m +\ \ K+1)$, then the configuration after the first round is of the form stated in Lemma 4.9, with $m_1 = m_2 = m$, $\ell_1 = 2$, $\ell_2 = \ \ K+1)$ and $k = \ \ K+1)$. Since $k = \ell_1$ (so that $k \in \{\ell_1, \ell_2\}$) and $\min\{\ell_1, \ell_2\} \neq 0$, hence P_2 wins by Lemma 4.9.

(f) If $w_1\ CD) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell \in \{0, 1, \ \ \ \ , 2^{R+1}\ \ \ \ 1\}$, the configuration is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 2$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\ \ B$, and a non-negative integer weight from BC, in the first round. We consider the following subcases:

(a) Suppose $w_1\ \ \ B) = 2^{R+1}m + \ell$ for some $\ell \in \{0,1\}$, and $w_1\ BC) = 1$. Then P_2 removes weight $\ 2+\ell)$ from the edge CD in the second round, leaving P_1 with

$$w_2\ \ \ B) = 2^{R+1}m + \ell,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m +\ \ K\ \ \ \ell),\ w_2\ EF) = \ \ K+1),$$

which is of the form given by (4.4) with either $r = 0$ (which happens when $\ell = 0$) or with $r = 1$ (which happens when $\ell = 1$), and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(b) Suppose $w_1\ \ \ B) = 2^{R+1}m + \ell$ for some $\ell \in \{0,1\}$, and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges $\ \ B, CD$ and EF, where $w_1\ \ \ B) = 2^{R+1}m + \ell$, $w_1\ CD) = 2^{R+1}m +\ \ K+2)$ and $w_1\ EF) = \ \ K+1)$. Since $K \equiv 0 \bmod 4$, the triple $\ 2^{R+1}m+1, 2^{R+1}m+K, K+1)$ is balanced when $\ell = 1$, and the triple $\ 2^{R+1}m, 2^{R+1}m+K+1, K+1)$ is balanced when $\ell = 0$. In either case, therefore, the triple $\ 2^{R+1}m+\ell, 2^{R+1}m+\ \ K+2), K+1)$ is not balanced, and therefore, P_2 wins by Theorem 2.3.

(c) If $w_1(AB) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell \in \{0, 1, \ldots, 2^{R+1} - 1\}$, the configuration is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = K+2$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = k \leqslant K$ (this implies $f(k) \leqslant f(K+1) = R$). If $k \geqslant 1$ and $k \equiv 1 \bmod 4$, then P_2 removes weight $(K+2) - (k+1)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m + 2,\ w_2(BC) = 1,\ w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m + (k+1),\ w_2(EF) = k,$$

which is of the same form as (4.5) with $r = 2$ and $s = 1$, and as $w_2(EF) = k \leqslant K$, we conclude that P_1 loses by our induction hypothesis. On the other hand, if $k \geqslant 6$ and $k \equiv i \bmod 4$ for some $i \in \{0, 2, 3\}$, then P_2 removes weight $(K+2) - (k-3)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m + 2,\ w_2(BC) = 1,\ w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m + (k-3),\ w_2(EF) = k,$$

which is of the same form as (4.6) with $r = 2$ and $s = 1$, and as $w_2(EF) = k \leqslant K$, we conclude that P_1 loses by our induction hypothesis. If $k = 2$, then the configuration after the first round is of the form given by Lemma 4.9, with $\ell_1 = 2$, $\ell_2 = (K+2)$ and $k = \ell_1$, which implies that $k \in \{\ell_1, \ell_2\}$. Hence, P_2 wins by Lemma 4.9. If $k = 3$, then, letting $K = 4n$ for some $n \in \mathbb{N}$ (since $K \equiv 0 \bmod 4$), the configuration obtained after the first round can be written as follows:

$$w_1(AB) = 2^{R+1}m + 2 = 4m_1 + 2, \text{ where } m_1 = 2^{R-1}m,$$
$$w_1(CD) = 2^{R+1}m + (K+2) = 2^{R+1}m + 4n + 2 = 4m_2 + 2, \text{ where } m_2 = 2^{R-1}m + n,$$

along with $w_1(BC) = 1$ and $w_1(EF) = 3$. This is of the same form as the configuration described in Lemma 4.9, with $m_1 \neq m_2$, which leads to the conclusion that P_2 wins. If $k = 4$, then P_2 removes weight $(K+1)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m + 2,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + 1,\ w_2(EF) = 4,$$

which is of the same form as (4.4) with $r = 1$, and by what we have already proved in §8.5, we know that P_1 loses. Finally, if $k = 0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes the edge BC in the first round, without disturbing the edge-weights of AB and CD. This leaves P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m + 2$, $w_1(CD) = 2^{R+1}m + (K+2)$ and $w_1(EF) = (K+1)$. Since $K \equiv 0 \bmod 4$, the triple $(2^{R+1}m + 2, 2^{R+1}m + (K+2), K)$ is balanced, and consequently, the triple $(2^{R+1}m + 2, 2^{R+1}m + (K+2), (K+1))$, is unbalanced. Therefore, P_2 wins by Theorem 2.3.

This completes the proof of our claim that the configuration in (7.1) is losing when $K \equiv 0 \bmod 4$.

We now let $K \equiv i \bmod 4$ for some $i \in \{1, 2, 3\}$, with $K \geqslant 6$, and consider the configuration

$$w_0(AB) = 2^{R+1}m + 2,\ w_0(BC) = 1,\ w_0(CD) = 2^{R+1}m + (K-2),\ w_0(EF) = (K+1), \tag{7.2}$$

where we set $R = f(K+1)$. The first round of the game played on this configuration can unfold as follows:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. This leads to a few possible subcases:

(a) Suppose $w_1(CD) = 2^{R+1}m + \ell$ for some $\ell \in \{2, 3, \ldots, K-3\}$, and $w_1(BC) = 1$. If $\ell \equiv j \bmod 4$ for some $j \in \{0, 1, 3\}$, then P_2 removes weight $(K+1) - (\ell+3)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell+3)+1}2^{R-f(\ell+3)}m + 2,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(\ell+3)+1}2^{R-f(\ell+3)}m + \ell,\ w_2(EF) = (\ell+3)$$

This configuration is of the same form as (4.6) with $r = 2$, $s = 1$ and $w_2\ EF) = \ \ell + 3) \leqslant K$, so that P_1 loses by our induction hypothesis. If $\ell \equiv 2 \bmod 4$, then P_2 removes weight $K+1) \quad \ell \quad 1)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \ \ell \quad 1)+1} 2^{R \quad f \ \ell \quad 1)} m + 2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f \ \ell \quad 1)+1} 2^{R \quad f \ \ell \quad 1)} m + \ell,\ w_2\ EF) = \ \ell \quad 1)$$

This configuration is of the same form as (4.5) with $r = 2$, $s = 1$ and $w_2\ EF) = \ \ell \quad 1) \leqslant K \quad 4) < K$, so that P_1 loses by our induction hypothesis. If $\ell \in \{0, 1\}$, then P_2 removes weight $K+1) \quad \ell + 3)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1} m + 2,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1} m + \ell,\ w_2\ EF) = \ \ell + 3),$$

which is of the form given by (4.4) with $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$), and by what we have already proved in §7.4 and §8.5, we know that P_1 loses.

(b) Suppose $w_1\ CD) = 2^{R+1} m + \ell$ for some $\ell \in \{0, 1, \quad , K \quad 3\}$ and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1} m + 2$, $w_1\ CD) = 2^{R+1} m + \ell$ and $w_1\ EF) = \ K+1)$. Since $\ell \leqslant \ K \quad 3) \implies \ 2 + \ell) \leqslant \ K \quad 1) < \ K+1)$, P_2 wins by Lemma 4.13.

(c) If $w_1\ CD) = 2^{R+1} n + \ell_1$ for some $n < m$ and $\ell \in \{0, 1, \quad , 2^{R+1} \quad 1\}$, the configuration is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 2$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from B, and a non-negative integer weight from CD, in the first round. Once again, we consider a few subcases:

(a) Suppose $w_1 \quad B) = 2^{R+1} m + \ell$ for some $\ell \in \{0, 1\}$, and $w_1\ BC) = 1$. Then P_2 removes weight $2 \quad \ell)$ from the edge EF in the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{R+1} m + \ell,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1} m + \ K \quad 2),\ w_2\ EF) = \ K \quad 1 + \ell),$$

which is of the form given by (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$), and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(b) Suppose $w_1 \quad B) = 2^{R+1} m + \ell$ for some $\ell \in \{0, 1\}$, and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1} m + \ell$, $w_1\ CD) = 2^{R+1} m + \ K \quad 2)$ and $w_2\ EF) = \ K+1)$, and P_2 wins by Lemma 4.13.

(c) If $w_1 \quad B) = 2^{R+1} n + \ell_1$ for some $n < m$ and $\ell \in \{0, 1, \quad , 2^{R+1} \quad 1\}$, the configuration is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = K \quad 2$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$:

(a) When $k \in \{K \quad 1, K\}$, P_2 removes weight $K+1 \quad k)$ from the edge B in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1} m + k + 1 \quad K,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1} m + \ K \quad 2),\ w_2\ EF) = k,$$

which is of the form given by (4.4) with $r = 0$ when $k = K \quad 1$, and of the form given by (4.4) with $r = 1$ when $k = K$, and by what we have already proved in §7.4 and §8.5, we know that P_1 loses. If $k = \ K \quad 2)$, the configuration after the first round is of the form given by Lemma 4.9, with $m_1 = m_2 = 2^{R \quad f\ K \quad 2)} m$, $\ell_1 = 2$ and $\ell_2 = \ K \quad 2)$, so that $k \in \{\ell_1, \ell_2\}$, and hence, P_2 wins by Lemma 4.9.

(b) Suppose $k \in \{1, 2, \quad , K \quad 3\}$, such that $k \equiv 1 \bmod 4$. Since $K \equiv i \bmod 4$ for some $i \in \{1, 2, 3\}$, we have $K \quad 3) \equiv j \bmod 4$ for some $j \in \{0, 2, 3\}$. Thus, this case does not cover $k = \ K \quad 3)$, and we may as well consider all $k \in \{1,2, \quad ,K \quad 4\}$ such that $k \equiv 1 \bmod 4$. Here, P_2 removes weight $\; K \quad 2) \quad k+1)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \; k)+1}2^{R \quad f \; k)}m+2,\; w_2 \; BC) = 1,$$
$$w_2 \; CD) = 2^{f \; k)+1}2^{R \quad f \; k)}m+ \; k+1),\; w_2 \; EF) = k,$$

which is of the form given by (4.5) with $r = 2$, $s = 1$ and $w_2 \; EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis.

(c) Suppose $k \in \{6,7, \quad ,K \quad 3\}$ such that $k \equiv j \bmod 4$ for some $j \in \{0,2,3\}$. In this case, P_2 removes weight $\; K \quad 2) \quad k \quad 3)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \; k)+1}2^{R \quad f \; k)}m+2,\; w_2 \; BC) = 1,$$
$$w_2 \; CD) = 2^{f \; k)+1}2^{R \quad f \; k)}m+ \; k \quad 3),\; w_2 \; EF) = k,$$

which is of the form given by (4.6) with $r = 2$, $s = 1$ and $w_2 \; EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis.

(d) Suppose $k \in \{2,3,4\}$. If $k = 2$, then the configuration after the first round is of the same form as in Lemma 4.9, with $m_1 = m_2 = 2^{R \quad 1}m$ (since $2^{f \; 2)+1} = 4$), $\ell_1 = 2$ and $\ell_2 = \; K \quad 2)$, so that $k \in \{\ell_1, \ell_2\}$, and hence, P_2 wins by Lemma 4.9. If $k \in \{3,4\}$, P_2 removes weight $\; K+1 \quad k)$ from CD during the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{R+1}m+2,\; w_2 \; BC) = 1,\; w_2 \; CD) = 2^{R+1}m+ \; k \quad 3),\; w_2 \; EF) = k,$$

which is of the form given by (4.4) with $r = 0$ when $k = 3$, and of the form given by (4.4) with $r = 1$ when $k = 4$, and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(iv) Suppose P_1 removes the edge BC in the first round, without perturbing the edge-weights of $\quad B$ and CD. This leaves P_2 with a galaxy graph consisting of the edges $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m+2$, $w_1 \; CD) = 2^{R+1}m+ \; K \quad 2)$ and $w_1 \; EF) = \; K+1)$, and P_2 wins by Lemma 4.13.

This completes the proof of our claim that (7.2) is a losing configuration on H_1. This also brings us to the end of our inductive proof of the claim that each of (4.5) and (4.6), with $r = 2$, represents a losing configuration on H_1.

7.7. **Proof that a configuration that is of the form given by** (4.7)**, with** $r = 2$**, is losing.** The base case for the inductive argument employed for this proof has been addressed in §8.7 of §8. Suppose, now, that for some $K \in \quad$ with $K \geqslant 6$, we have shown that a configuration of the form given by (4.7), with $r = 2$, is losing as long as $w_0 \; EF) = k \leqslant K$. We now consider the configuration (setting $s = \; K \quad L \quad 1)$):

$$w_0 \quad B) = 2^{R+1}m+2,\; w_0 \; BC) = K \quad L \quad 1,\; w_0 \; CD) = 2^{R+1}m+L,\; w_0 \; EF) = K+1, \tag{7.3}$$

where $R = f \; K+1)$ and $L \in \{2,3, \quad ,K \quad 3\}$. The first round of the game played on this configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. First, we consider the possibility that $w_1 \; CD) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1, \quad ,L \quad 1\}$ and $w_1 \; BC) = t \in \{0,1, \quad ,K \quad L \quad 1\}$. This can be divided into a few subcases:

(a) Suppose $\ell \in \{0,1\}$ and $t \in \{1,2, \quad ,K \quad L \quad 1\}$, or $\ell \in \{2,3, \quad ,L \quad 1\}$ and $t \in \{2,3, \quad ,K \quad L \quad 1\}$. In each of these cases, P_2 removes weight $\; K+1) \quad \ell+t+2)$ from the edge EF in the second round. Note that $\ell \leqslant \; L \quad 1)$ and $t \leqslant \; K \quad L \quad 1)$ together imply $\; \ell+t+2) \leqslant K$, which, in turn, implies $f \; \ell+t+2) \leqslant f \; K+1) = R$. This leaves P_1 with

$$w_2 \quad B) = 2^{f \; \ell+t+2)+1}2^{R \quad f \; \ell+t+2)}m+2,\; w_2 \; BC) = t,$$

$$w_2\ CD) = 2^{f\ \ell+t+2)+1}2^{R\ \ f\ \ell+t+2)}m+\ell,\ w_2\ EF) = \ \ \ell+t+2),$$

which is of the form given by (4.4) with $r = 0$ when $\ell = 0$, of the form given by (4.4) with $r = 1$ when $\ell = 1$, and of the form given by (4.7) with $r = 2$ and $w_2\ EF) = \ \ \ell+t+2) \leqslant K$ when $\ell \in \{2,3,\ \ \ ,L\ \ \ 1\}$ and $t \in \{2,3,\ \ \ ,K\ \ \ L\ \ \ 1\}$. In the first two cases, P_1 loses by what we have already proved in §7.4 and §8.5, while in the third case, P_1 loses by our induction hypothesis.

(b) Suppose $\ell \in \{2,3,\ \ \ ,L\ \ \ 1\}$ and $t = 1$. If $\ell \equiv 2 \bmod 4$, then P_2 removes weight $\ \ K+1)$ $\ \ \ell\ \ \ 1)$ from the edge EF in the second round, leaving P_1 with

$$w_2\ \ \ B) = 2^{f\ \ell\ \ \ 1)+1}2^{R\ \ f\ \ell\ \ \ 1)}m+2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ \ell\ \ \ 1)+1}2^{R\ \ f\ \ell\ \ \ 1)}m+\ell,\ w_2\ EF) = \ \ \ell\ \ \ 1),$$

which is of the same form as (4.5) with $r = 2$ and $s = 1$, and by what we have already proved in §7.6, we conclude that P_1 loses. If $\ell \equiv i \bmod 4$ for some $i \in \{0,1,3\}$, then P_2 removes $\ K+1)\ \ \ \ \ \ell+3)$ from the edge EF in the second round (note that $\ell \leqslant L\ \ \ 1$ and $L \leqslant K\ \ \ 3$ together imply that $\ell+3 \leqslant K\ \ \ 1$), leaving P_1 with

$$w_2\ \ \ B) = 2^{f\ \ell+3)+1}2^{R\ \ f\ \ell+3)}m+2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ \ell+3)+1}2^{R\ \ f\ \ell+3)}m+\ell,\ w_2\ EF) = \ \ \ell+3),$$

which is of the same form as (4.6) with $r = 2$ and $s = 1$, and by what we have already proved in §7.6, we know that P_1 loses.

(c) If $\ell \in \{0,1,\ \ \ ,L\ \ \ 1\}$ and $t = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges $\ \ B, CD$ and EF, where $w_1\ \ \ B) = 2^{R+1}m+2$, $w_1\ CD) = 2^{R+1}m+\ell$ and $w_1\ EF) = \ \ K+1)$. Since $\ell \leqslant\ \ L\ \ \ 1)$ and $L \leqslant\ \ K\ \ \ 3)$, we conclude, by Lemma 4.13, that P_2 wins.

The second possibility is that $w_1\ CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1,\ \ \ ,2^{R+1}\ \ \ 1\}$. In this case, the configuration after the first round is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 2$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\ \ B$, and a non-negative integer weight from BC, in the first round. The first possibility is where $w_1\ \ \ B) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1\}$, and $w_1\ BC) = t \in \{0,1,\ \ \ ,K\ \ \ L\ \ \ 1\}$. We divide this into a few subcases:

(a) If $t \in \{1,2,\ \ \ ,K\ \ \ L\ \ \ 1\}$, then P_2 removes weight $\ K+1)\ \ \ \ \ \ell+t+L)$ from the edge EF in the second round. Note that $t \leqslant K\ \ \ L\ \ \ 1$ and $\ell \leqslant 1$ together imply that $\ \ \ell+t+L) \leqslant K$. This leaves P_1 with

$$w_2\ \ \ B) = 2^{f\ \ell+t+L)+1}2^{R\ \ f\ \ell+t+L)}m+\ell,\ w_2\ BC) = t,$$
$$w_2\ CD) = 2^{f\ \ell+t+L)+1}2^{R\ \ f\ \ell+t+L)}m+L,\ w_2\ EF) = \ \ \ell+t+L),$$

which is of the same form as (4.4) with either $r = 0$ when $\ell = 0$ or $r = 1$ when $\ell = 1$, and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(b) If $t = 0$, then P_2, after the first round, is left with a galaxy graph consisting of the edges $\ \ B$, CD and EF, where $w_1\ \ \ B) = 2^{R+1}m+\ell$, $w_1\ CD) = 2^{R+1}m+L$ and $w_1\ EF) = \ \ K+1)$. Since $\ell \leqslant 1$ and $L \leqslant\ \ K\ \ \ 3)$, we conclude, by Lemma 4.13, that P_2 wins.

The second possibility is that $w_1\ \ \ B) = 2^{R+1}n+\ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1,\ \ \ ,2^{R+1}\ \ \ 1\}$. In this case, the configuration after the first round is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = L$, and as $n < m \implies m_1 \neq m_2$, we know that P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round. If $w_1\ EF) = k \in \{K\ \ L+1,\ \ ,K\}$, P_2 removes weight $\ K+1\ \ k)$ from CD in the second round, leaving P_1 with

$$w_2\ \ B) = 2^{f\ k)+1}2^{R\ \ f\ k)}m+2,\ w_2\ BC) = K\ \ L\ \ 1,$$
$$w_2\ CD) = 2^{f\ k)+1}2^{R\ \ f\ k)}m+L+k\ \ K\ \ 1,\ w_2\ EF) = k,$$

which is of the form given by (4.4) with $r = 0$ when $k = K\ \ L+1$, of the form given by (4.4) with $r = 1$ when $k = K\ \ L+2$, and of the form given by (4.7) with $r = 2$ and $w_2\ EF) = k \leqslant K$ when $k \in \{K\ \ L+3,\ \ ,K\}$. That P_1 loses in the first two cases follows from what we have already proved in §7.4 and §8.5, and that she loses in the third case follows from our induction hypothesis. If $k \in \{1,2,\ \ ,K\ \ L\}$, then the configuration after the first round is of the same form as in Lemma 4.11, with $\ell_1 = 2$, $\ell_2 = L$ and $w_1\ BC) = \ \ K\ \ L\ \ 1) > w_1\ EF)\ \ \min\{\ell_1,\ell_2\}$, so that P_2 wins by Lemma 4.11. If $k = 0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of $\ B$ and CD, so that $w_1\ BC) = t \in \{0,1,\ \ ,K\ \ L\ \ 2\}$.

(a) If $t \in \{2,3,\ \ ,K\ \ L\ \ 2\}$, P_2 removes weight $\ K+1)\ \ \ 2+t+L)$ from the edge EF in the second round. Note that $\ 2+t+L) \leqslant 2+\ \ K\ \ L\ \ 2)+L = K$. This leaves P_1 with

$$w_2\ \ B) = 2^{f\ 2+t+L)+1}2^{R\ \ f\ 2+t+L)}m+2,\ w_2\ BC) = t,$$
$$w_2\ CD) = 2^{f\ 2+t+L)+1}2^{R\ \ f\ 2+t+L)}m+L,\ w_2\ EF) = \ 2+t+L),$$

which is of the same form as (4.7) with $r = 2$ and $w_2\ BC) = t \geqslant 2$, and as $w_2\ EF) = \ 2+t+L) \leqslant K$, we conclude that P_1 loses by our induction hypothesis.

(b) Suppose $t = 1$. Here, if $L \equiv 2 \bmod 4$, then P_2 removes weight $\ K+1)\ \ \ L\ \ 1)$ from the edge EF in the second round, leaving P_1 with

$$w_2\ \ B) = 2^{f\ L\ \ 1)+1}2^{R\ \ f\ L\ \ 1)}m+2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L\ \ 1)+1}2^{R\ \ f\ L\ \ 1)}m+L,\ w_2\ EF) = \ L\ \ 1),$$

which is of the form given by (4.5) with $r = 2$ and $w_1\ BC) = s = 1$, and by what we have already proved in §7.6, we conclude that P_1 loses. If, on the other hand, $L \equiv i \bmod 4$ for some $i \in \{0,1,3\}$, then P_2 removes weight $\ K+1)\ \ \ L+3)$ from the edge EF in the second round (recall that $L \leqslant\ K\ \ 3) \implies\ K+1)\ \ \ L+3) > 0$), leaving P_1 with

$$w_2\ \ B) = 2^{f\ L+3)+1}2^{R\ \ f\ L+3)}m+2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L+3)+1}2^{R\ \ f\ L+3)}m+L,\ w_2\ EF) = \ L+3),$$

which is of the form given by (4.6) with $r = 2$ and $w_1\ BC) = s = 1$, and by what we have already proved in §7.6, we know that P_1 loses.

(c) Finally, if $t = 0$, P_2 is left with a galaxy graph at the end of the first round, consisting of the edges $\ B$, CD and EF, where $w_1\ \ B) = 2^{R+1}m+2$, $w_1\ CD) = 2^{R+1}m+L$ and $w_1\ EF) = \ K+1)$. Since $L \leqslant\ K\ \ 3)$, we conclude, by Lemma 4.13, that P_2 wins.

This concludes the proof of our claim that the configuration in (7.3) is losing. This also concludes the inductive proof of our claim that any configuration on H_1 of the form given by (4.7), with $r = 2$, is losing.

8. PPENDIX

The appendix section of this paper has been dedicated to showcasing all those details whose inclusion in the main body of the paper would make reading and comprehension of the proof techniques cumbersome and prove a hindrance to lucidity.

8.1. **Details omitted from the proof of Theorem 4.3.** We prove the base case of the inductive argument employed in the proof of Theorem 4.3, for proving that the configuration in (5.1) is losing. Since the induction happens with respect to the edge-weight $w_0\ CD)$, the base case consists of $w_0\ \ B) = w_0\ BC) = w_0\ DB) = 1$ and $w_0\ CD) = 2$. We consider the first round of the game played on this initial configuration:

(i) Suppose P_1 selects the vertex B in the first round, and removes at least one of the edges B, BC and DB.
   (a) If P_1 removes only B, then in the second round, P_2 removes weight 1 from CD (by choosing either C or D), leaving behind a triangle with all three edge-weights equal, so that P_1 loses by Theorem 5.1.
   (b) If P_1 removes only BC (analogously, P_1 removes only DB), then P_2 selects D and removes weight 1 from each of DB and CD in the second round (analogously, P_2 selects C and removes weight 1 from each of BC and CD in the second round), leaving P_1 with B and CD where $w_0\ \ B) = w_0\ CD) = 1$, so that P_1 loses by Theorem 2.3.
   (c) If P_1 removes both B and BC (which is analogous to her removing both B and DB), then P_2 selects D and removes both DB and CD in the second round (analogously, she selects C and removes both BC and CD), thus winning the game immediately.
   (d) If P_1 removes both BC and DB, then P_2 removes weight 1 from CD in the second round, so that P_1 loses by Theorem 2.3.
   (e) If P_1 removes all three of B, BC and DB, then P_2 removes CD in the second round, thus winning the game immediately.

(ii) Suppose P_1 selects the vertex C in the first round (which is analogous to her selecting the vertex D). If P_1 removes only BC, then P_2's response is the same as the corresponding case discussed above.
   (a) If P_1 removes only weight 1 from CD, and none from BC, then P_2 removes the edge B in the second round, and P_1 loses by Theorem 5.1.
   (b) If P_1 removes the entire edge CD, but leaves BC intact, then P_2 selects the vertex B and removes all of B, DB and BC in the second round, thus winning the game immediately.
   (c) If P_1 removes weight 1 from each of BC and CD, then P_2 removes the edge DB in the second round, and P_1 loses by Theorem 2.3.
   (d) If P_1 removes both BC and CD, then P_2 selects the vertex B and removes both B and DB in the second round, thus winning the game immediately.

(iii) Suppose P_1 selects , and removes B in the first round. Then P_2 removes weight 1 from CD in the second round, and wins by Theorem 5.1.

This shows us that no matter the move made by P_1 in the first round, P_2 possesses a winning strategy, thus completing the proof of the base case for the inductive argument employed in proving that any configuration on F_2 that is of the form (5.1) is losing.

8.2. **Details omitted from the proof of Step** (D2) **meant for proving Theorem 4.6.** We prove that a configuration satisfying the hypothesis of (1) is losing when $m = 1$, $k = 1$, $i = 1$ and $\ell \in\ \ _0$, i.e. we show that the following configuration is losing on G_4:

$$w_0\ \ B) = w_0\ BC) = 2 + 2\ell, \quad w_0\ C\ \ ) = 3 + 2\ell \quad \text{and} \quad w_0\ DE) = 1 \tag{8.1}$$

To begin with, we consider the configuration obtained from (8.1) by setting $\ell = 0$, i.e. where $w_0\ \ B) = w_0\ BC) = 2$, $w_0\ C\ \ ) = 3$ and $w_0\ DE) = 1$. In the first round of the game played on this initial weight configuration, one of the following transpires:

(i) Suppose P_1 removes edge-weights from at most two of the edges among B, BC and C , leaving P_2 with $\min\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\} \in \{0, 1, 2\}$. Note that this minimum equals 2 if and only if P_1 removes weight 1 from the edge C in the first round, and leaves all else undisturbed, so that in this case, it suffices for P_2 to simply remove the edge DE in the second round, leaving P_1 with B, BC and C , with $w_2\ \ B) = w_2\ BC) = w_2\ C\ \ ) = 2$. Therefore, P_1 loses by Theorem 5.1.

If $\min\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\} = 1$, then P_2 removes all edges, out of B, BC and C , except for the one whose edge-weight after the first round of the game equals 1. This leaves P_1 with two disjoint edges, one of which is DE, each having edge-weight 1, and P_1 loses by Theorem 2.3.

If $\min\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\} = 0$, then at least one of the edges out of B, BC and C has been completely removed in the first round. Note that this already leaves P_2 with a galaxy graph whose connected components are two star graphs: one of which consists of one or two edges out of B, BC and C , and the other consists of only the edge DE. Since P_1 can alter the edge-weights of at most two of the edges B, BC and C in the first round, the former of these two components has at least one ray with edge-weight strictly greater than 1. P_2, therefore, can remove the requisite weight from the former component in the second round, so that P_1 is left with two disjoint edges (one of which is DE), each with edge-weight 1, and she loses by Theorem 2.3.

(ii) Suppose P_1 removes the edge DE in the first round. Then P_2 removes weight 1 from the edge C in the second round, leaving P_1 with B, BC and C where $w_2\ \ B) = w_2\ BC) = w_2\ C\ \ ) = 2$. Consequently, P_1 loses by Theorem 5.1.

This completes the proof of our claim that the configuration in (8.1) is losing for $\ell = 0$.

Suppose we have proved that the configuration in (8.1) is losing whenever $\ell \leqslant L$, for some $L \in\ \ _0$. We now consider the configuration

$$w_0\ \ B) = w_0\ BC) = 2+2\ L+1), \quad w_0\ C\ \ ) = 3+2\ L+1) \quad \text{and} \quad w_0\ DE) = 1 \tag{8.2}$$

The first round of the game played on this initial weight configuration unfolds in one of the following ways:

(i) Suppose P_1 removes edge-weights from at most two of the edges B, BC and C in the first round, such that

$$\min\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\} = 2\ i+1) \text{ for some } i \in\ \ _0$$

Note that we must have $2\ i+1) \leqslant 2+2\ L+1) = 2\ L+2) \implies i \leqslant\ L+1)$. If $i =\ L+1)$, then P_1 must have removed weight 1 from the edge C , and left all else unchanged, in the first round. However, this means that $w_1\ \ B) = w_1\ BC) = w_1\ C\ \ ) = 2\ L+2)$, and P_2 simply removes the edge DE in the second round so as to make P_1 lose by Theorem 5.1.

Suppose $i \leqslant L$. In this case, $3+2i \leqslant 3+2L < 4+2L = 2+2\ L+1)$. Therefore, P_2, in the second round, can select an appropriate vertex out of , B and C, and reduce the edge-weights of the two edges incident on it such that the edge-weights of B, BC and C , *in some order*, become $2+2i)$, $2+2i)$ and $3+2i)$. s an example, if P_1 selects the vertex C and removes weights from BC and C in the first round in such a manner that

$$w_1\ C\ \ ) = 2\ i+1) = 2+2i \quad \text{and} \quad 2+2i \leqslant w_1\ BC) \leqslant w_0\ BC),$$

then in the second round, P_2 selects the vertex B, removes edge-weight $w_1\ BC)\quad 2+2i)$ from BC, and removes edge-weight $w_1\ \ B)\quad 3+2i) = w_0\ \ B)\quad 3+2i) = \{2+2\ L+1)\}\quad 3+2i)$ from B, so that P_1 is left with

$$w_2\ BC) = w_2\ C\ \ ) = 2+2i, \quad w_2\ \ B) = 3+2i \quad \text{and} \quad w_2\ DE) = 1,$$

which is of the same form as (8.1), with $\ell = i \leqslant L$. Consequently, P_1 loses by our induction hypothesis.

If, on the other hand, $\min\{w_1\ \ B), w_1\ BC), w_1\ C\ \ )\} = 0$, then, after the first round, P_2 is left with a galaxy graph with two components, one of which is the edge DE with edge-weight 1, while the other has at least one ray whose edge-weight is at least $2+2\ L+1)$. It is evident from Theorem 2.3 that P_2 wins.

(ii) Suppose P_1 removes edge-weights from at most two of the edges $\ \ B$, BC and $C\ \ $ in the first round, such that

$$\min\{w_1\ \ \ B), w_1\ BC), w_1\ C\ \ )\} = 2\ \ i+1)+1 \text{ for some } i \in\ \ _0$$

Note that we must have $2\ \ i+1)+1 < 2+2\ \ L+1) \implies i \leqslant L$, which, in turn, implies that $2+2i \leqslant 2+2L < 2+2\ \ L+1)$. In this case, P_2 selects an appropriate vertex out of $\ \ B$, BC and $C\ \ $, and reduces the edge-weights of the two edges incident on it such that the edge-weights of $\ \ B$, BC and $C\ \ $, at the end of the second round, become $\ \ 2+2i)$, $\ \ 2+2i)$ and $2\ \ i+1)+1 = \ \ 3+2i)$ *in some order*. For instance, if P_1 selects the vertex B and removes weights from $\ \ B$ and BC in the first round in such a manner that

$$w_1\ \ \ B) = 2\ \ i+1)+1 \quad \text{and} \quad 2\ \ i+1)+1 \leqslant w_1\ BC) \leqslant w_0\ BC),$$

then P_2, in the second round, selects the vertex C and removes weight $w_1\ BC)\ \ \ \ 2+2i)$ from the edge BC and weight $w_1\ C\ \ )\ \ \ \ 2+2i) = w_0\ C\ \ )\ \ \ \ 2+2i) = 3+2\ \ L+1)\ \ \ \ 2+2i)$ from the edge $C\ \ $. This leaves P_1 with

$$w_2\ \ \ B) = 2\ \ i+1)+1 = 3+2i, \quad w_2\ BC) = w_2\ C\ \ ) = 2+2i \quad \text{and} \quad w_2\ DE) = 1,$$

and as $i \leqslant L$, hence P_1 loses by our induction hypothesis.

On the other hand, if $\min\{w_1\ \ \ B), w_1\ BC), w_1\ C\ \ )\} = 1$, then P_2 removes, by choosing an appropriate vertex out of $\ $, B and C in the second round, two of the edges out of $\ \ B$, BC and $C\ \ $, leaving behind only the third that has edge-weight 1. This leaves P_1 with two disjoint edges (one of which is DE), each with edge-weight 1, and she loses by Theorem 5.1.

(iii) Finally, if P_1 removes the edge DE in the first round, then P_2 removes weight 1 from $C\ \ $ in the second, making P_1 lose by Theorem 5.1.

This concludes the proof of the fact that the configuration in (8.2) is losing, thus completing the inductive proof of the claim that the configuration in (8.1) is losing for all $\ell \in\ \ _0$. This establishes the base case of the inductive argument employed in proving Step (D2).

8.3. **Details omitted from the proof of Step** (D4) **meant for proving Theorem 4.6.** The base case for the inductive argument employed for proving Step (D4) corresponds to $w_0\ DE) = 4$, two of $w_0\ \ \ B)$, $w_0\ BC)$ and $w_0\ C\ \ )$ being equal to 1 each, and the third being equal to 2 (this is because we must ensure that $w_0\ \ \ B)$, $w_0\ BC)$ and $w_0\ C\ \ )$ are not all equal). Without loss of generality, let us consider $w_0\ \ \ B) = w_0\ BC) = 1$, $w_0\ C\ \ ) = 2$ and $w_0\ DE) = 4$.

The first round of the game played on this initial configuration can unfold in one of the following ways:

(i) Suppose P_1 removes, in the first round, at least one of the edges $\ \ B$, BC and $C\ \ $. Then P_2 is left with a galaxy graph with two components: one of which is the edge DE with $w_1\ DE) = 4$, and the other having the sum of its edge-weights strictly less than 4. Therefore, the corresponding configuration is unbalanced, and P_2 wins by Theorem 2.3.

(ii) Suppose P_1 removes weight 1 from the edge $C\ \ $ in the first round, and leaves all else unchanged. Then P_2 removes the edge DE in the second round, and wins by Theorem 5.1.

(iii) Suppose P_1 removes some edge-weight from DE in the first round. If $w_1\ DE) = 3$, then P_2 removes the edge $\ \ B$ in the second round, leaving P_1 with a galaxy graph comprising two components: one of which is the edge DE with $w_2\ DE) = 3$, and the other is a star graph consisting of the edges BC and $C\ \ $, with $w_2\ BC) + w_2\ C\ \ ) = 3$. Consequently, P_1 loses by Theorem 2.3. If $w_1\ DE) = 2$, then P_2 removes edges $\ \ B$ and BC in the second round, and if $w_1\ DE) = 1$, then P_2 removes edges BC and $C\ \ $ in the second round. In either of these cases, once again, P_1 loses by Theorem 2.3. If $w_1\ DE) = 0$, then P_2 removes weight 1 from $C\ \ $ in the second round, and P_1 loses by Theorem 5.1.

This completes the proof of the base case for the inductive argument employed in proving Step (D4).

8.4. **Proof of Lemma 4.13.** We set $R = f(k)$, and since each of ℓ_1 and ℓ_2 is in $\{0,1,\dots,2^{R+1}-1\}$, we can write the base-2 representations of ℓ_1 and ℓ_2 as $\ell_1 = (a_R a_{R-1} \dots a_1 a_0)_2$ and $\ell_2 = (b_R b_{R-1} \dots b_1 b_0)_2$. We now find $c_i \in \{0,1\}$, for each $i \in \{0,1,\dots,R\}$, such that $a_i + b_i + c_i$ is even. Evidently, the triple

$$\left(2^{R+1}m+\ell_1, 2^{R+1}m+\ell_2, \sum_{i=0}^{R} c_i 2^i\right)$$

is balanced, and therefore, losing by Theorem 2.1. Note that, for each $i \in \{0,1,\dots,R\}$,

(i) either at most one of a_i and b_i equals 1, in which case we have $c_i = a_i + b_i$,
(ii) or else $a_i = b_i = 1$, in which case $c_i = 0 < a_i + b_i$.

Combining these observations, we can write the inequality:

$$\sum_{i=0}^{R} c_i 2^i \leqslant \sum_{i=0}^{R} (a_i + b_i) 2^i = \ell_1 + \ell_2 < k$$

Consequently, the triple $(2^{R+1}m+\ell_1, 2^{R+1}m+\ell_2, k)$ must be unbalanced, and therefore, winning by Theorem 2.1. This completes the proof of Lemma 4.13.

8.5. **Proof that any configuration of the form given by** (4.4)**, with** $r = 1$**, is losing.** The proof that any configuration on H_1 that is of the form given by (4.4), with $r = 1$, is inductive, and we begin by establishing the base case, obtained by setting $s = 1$ and $k = 3$ in (4.4). Consider the first round of the game played on this initial configuration, i.e. on $w_0(AB) = w_0(CD) = 4m+1$, $w_0(BC) = 1$ and $w_0(EF) = 3$, for any $m \in \mathbb{N}$:

(i) Suppose P_1 removes a positive integer weight from AB and a non-negative integer weight from BC in the first round (an analogous situation would be where P_1 removes a positive integer weight from CD and a non-negative integer weight from BC). We consider a few subcases:
  (a) Suppose $w_1(AB) = 4m$ and $w_1(BC) = 1$. Then P_2 removes weight 1 from EF in the second round, leaving P_1 with $w_2(AB) = 4m$, $w_2(BC) = 1$, $w_2(CD) = 4m+1$ and $w_2(EF) = 2$, which is of the same form as (4.4) with $r = 0$, $s = 1$ and $k = 2$, and by what we have already proved in §7.4, we conclude that P_1 loses.
  (b) Suppose $w_1(AB) = 4m$ and $w_1(BC) = 0$. This leaves P_2 with a galaxy graph comprising the edges AB, CD and EF, where $w_1(AB) = 4m$, $w_1(CD) = 4m+1$ and $w_1(EF) = 3$, and it is evident, from Lemma 4.13, that this configuration is winning, allowing P_2 to defeat P_1.
  (c) Suppose $w_1(AB) = 4n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1,2,3\}$. Then the resulting configuration is of the same form as in Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 1$, and as $n < m \implies m_1 \neq m_2$, hence P_2 wins.

(ii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = t \in \{0,1,2\}$. If $t = 2$, P_2 removes weight 1 from CD in the second round, leaving P_1 with $w_2(AB) = 4m+1$, $w_2(BC) = 1$, $w_2(CD) = 4m$ and $w_2(EF) = 2$, which is of the form given by (4.4) with $r = 0$, $s = 1$ and $k = 2$, so that P_1 loses by what we have already proved. If $t = 1$, then P_2 removes weight 1 from each of BC and CD in the second round, leaving P_1 with a galaxy graph comprising the edges AB, CD and EF, where $w_2(AB) = 4m+1$, $w_2(CD) = 4m$ and $w_2(EF) = 1$, so that P_1 loses by Theorem 2.3. Finally, if $t = 0$, P_2 wins by Remark 4.12.

(iii) If P_1 removes BC in the first round, without disturbing the edge-weights of AB and CD, P_2 is left with a galaxy graph comprising the edges AB, CD and EF, with $w_1(AB) = w_1(CD) = 4m+1$ and $w_1(EF) = 3$, and P_2 wins by Lemma 4.13.

Suppose, for some $K \in \mathbb{N}$, we have shown that any configuration of the form given by (4.4), with $r = 1$ and $k \leqslant K$, is losing. We now consider the first round of the game played on the initial weight configuration

$$w_0(AB) = 2^{R+1}m+1,\ w_0(BC) = K-L,\ w_0(CD) = 2^{R+1}m+L,\ w_0(EF) = K+1, \tag{8.3}$$

where $R = f\ \ K+1)$ and $L \in \{1,2, \quad ,K \quad 1\}$:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. We subdivide this case into a few subcases, as follows:

(a) Suppose $w_1\ \ CD) = 2^{R+1}m + \ell$ for some $\ell \in \{0,1, \quad ,L \quad 1\}$, and $w_1\ \ BC) = t \in \{1,2, \quad ,K \quad L\}$. Note that $t + \ell + 1 \leqslant \ \ K \quad L) + \ \ L \quad 1) + 1 = K$. Here, P_2 removes weight $\ \ K \quad t \quad \ell)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ \ t+\ell+1)+1} 2^{R \quad f\ \ t+\ell+1)} m + 1,\ w_2\ \ BC) = t,$$
$$w_2\ \ CD) = 2^{f\ \ t+\ell+1)+1} 2^{R \quad f\ \ t+\ell+1)} m + \ell,\ w_2\ \ EF) = t + \ell + 1$$

When $\ell \geqslant 1$, this configuration is of the form given by (4.4) with $r = 1$ and $w_2\ \ EF) = t + \ell + 1 \leqslant K$, so that P_1 loses by our induction hypothesis; on the other hand, when $\ell = 0$, this configuration is of the form given by (4.4) with $r = 0$, and by what we have already proved in §7.4, we conclude that P_1 loses.

(b) Suppose $w_1\ \ CD) = 2^{R+1}m + \ell$ for some $\ell \in \{0,1, \quad ,L \quad 1\}$ and $w_1\ \ BC) = 0$. This leaves P_2 with a galaxy graph comprising the edges $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m + 1$, $w_1\ \ CD) = 2^{R+1}m + \ell$ and $w_1\ \ EF) = \ \ K+1)$. Since $\ell + 1 \leqslant L \leqslant \ \ K \quad 1) < \ \ K+1)$, it is immediate, from Lemma 4.13, that P_2 wins.

(c) Suppose $w_1\ \ CD) = 2^{R+1}n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$. The resulting configuration is of the form given by Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = 1$, and as $n < m \implies m_1 \neq m_2$, hence P_2 wins.

(ii) Suppose P_1 removes a positive integer weight from $\quad B$ and a non-negative integer weight from BC in the first round. Once again, we subdivide our analysis as follows:

(a) If $w_1 \quad B) = 2^{R+1}m$ and $w_1\ \ BC) = t \in \{1,2, \quad ,K \quad L\}$, P_2 removes $\ \ K+1) \quad \ \ t+L)$ from EF in the second round. Note that $\ \ t+L) \leqslant \ \ K \quad L) + L = K$. This leaves P_1 with

$$w_2 \quad B) = 2^{f\ \ t+L)+1} 2^{R \quad f\ \ t+L)} m,\ w_2\ \ BC) = t,$$
$$w_2\ \ CD) = 2^{f\ \ t+L)+1} 2^{R \quad f\ \ t+L)} m + L,\ w_2\ \ EF) = t + L,$$

which is of the same form as (4.4) with $r = 0$, and by what we have already proved in §7.4, we conclude that P_1 loses.

(b) If $w_1 \quad B) = 2^{R+1}m$ and $w_1\ \ BC) = 0$, P_2 is left with a galaxy graph comprising the edges $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m$, $w_1\ \ CD) = 2^{R+1}m + L$ and $w_1\ \ EF) = \ \ K+1)$, and as $L \leqslant \ \ K \quad 1) < \ \ K+1)$, P_2 wins by Lemma 4.13.

(c) Suppose $w_1 \quad B) = 2^{R+1}n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$. The resulting configuration is of the form given by Lemma 4.9, with $m_1 = n$, $m_2 = m$ and $\ell_2 = L$, and as $n < m \implies m_1 \neq m_2$, hence P_2 wins.

(iii) Suppose P_1 removes a positive integer weight from EF, so that $w_1\ \ EF) = k \leqslant K$. When $k \in \{K \quad L+1, \quad ,K\}$, P_2 removes weight $L \quad \{k \quad \ K \quad L) \quad 1\}$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ \ k)+1} 2^{R \quad f\ \ k)} m + 1,\ w_2\ \ BC) = \ \ K \quad L),$$
$$w_2\ \ CD) = 2^{f\ \ k)+1} 2^{R \quad f\ \ k)} m + k \quad \ K \quad L) \quad 1,\ w_2\ \ EF) = k$$

When $k \geqslant K \quad L+2$, this configuration is of the form given by (4.4) with $r = 1$, and P_1 loses by our induction hypothesis since $k \leqslant K$; on the other hand, when $k = K \quad L+1$, it is of the form given by (4.4) with $r = 0$, and by what we have already proved in §7.4, we conclude that P_1 loses. If $k \in \{1,2, \quad ,K \quad L \quad 1\}$, the configuration after the first round is of the form stated in Lemma 4.11, with $w_1\ \ BC) > w_1\ \ EF)$, and P_2 wins by Lemma 4.11. If $k = K \quad L$, the configuration after the first round is of the form stated in Lemma 4.11, with $\ell_1 = 1$, $\ell_2 = L$, and $w_1(BC) = (K-L) > (K-L) - 1 = w_1(EF) \geqslant \min\{\ell_1, \ell_2\}$. Finally, if $k = 0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from BC in the first round, leaving the edge-weights of AB and CD undisturbed. As long as $w_1(BC) = t \in \{1, 2, \ldots, K-L-1\}$, P_2 removes $(K-L-t)$ from the edge EF in the second round, leaving P_1 with a configuration that is of the same form as (4.4) with $r = 1$ and $w_2(EF) = L + t + 1 \leqslant K$, so that P_1 loses by our induction hypothesis.

If $w_1(BC) = 0$, then P_2 is left with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m + 1$, $w_1(CD) = 2^{R+1}m + L$ and $w_1(EF) = (K+1)$. Since $L \leqslant (K-1)$, we conclude, by Lemma 4.13, that P_2 wins.

This concludes the proof of our claim that the configuration in (8.3) is losing, thus completing the inductive proof of our claim that any configuration on H_1 of the form given by (4.4) with $r = 1$, is losing.

8.6. **Details omitted from our proof that any configuration that is either of the form** (4.5) **or of the form** (4.6)**, with** $r = 2$**, is losing.** We include here the details omitted from the proof provided in §7.6. To begin with, we consider the base case for the inductive argument used to prove that (4.5), for $r = 2$, is losing, which is obtained by setting $k = 1$ in (4.5) (note that since $r = 2$, we must have $s = 1$). This yields the configuration (for any $m \in \mathbb{N}$):

$$w_0(AB) = w_0(CD) = 2m + 2 = 2(m+1),\ w_0(BC) = w_0(EF) = 1,$$

which is of the same form as (4.4) with $r = 0$, and by what we have already proved in §7.4, we know it must be losing. The base case for the inductive argument used to prove that (4.6), with $r = 2$, is losing, is obtained by setting $k = 6$ in (4.6) (once again, since $r = 2$, we must have $s = 1$), which yields the configuration (for any $m \in \mathbb{N}$):

$$w_0(AB) = 8m + 2,\ w_0(BC) = 1,\ w_0(CD) = 8m + 3,\ w_0(EF) = 6 \tag{8.4}$$

We consider the first round of the game played on this initial configuration:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. We consider a few subcases of this case:

(a) Suppose $w_1(CD) = 8m + 2$ and $w_1(BC) = 1$. Then P_2 removes weight 5 from EF, so that P_1 is left with

$$w_2(AB) = w_2(CD) = 8m + 2 = 2(4m+1),\ w_2(BC) = w_2(EF) = 1,$$

which is of the same form as (4.4) with $r = 0$, and by what we have already proved in §7.4, we conclude that P_1 loses. Suppose $w_1(CD) = 8m + i$ for $i \in \{0, 1\}$, and $w_1(BC) = 1$. Then P_2 removes weight $(3 - i)$ from EF in the second round, so that P_1 is left with

$$w_2(AB) = 8m + 2,\ w_2(BC) = 1,\ w_2(CD) = 8m + i,\ w_2(EF) = 3 + i,$$

which is of the form given by (4.4) with either $r = 1$ (when $i = 1$) or $r = 0$ (when $i = 0$), and by what we have already proved in §7.4 and §8.5, we know that P_1 loses.

(b) Suppose $w_1(CD) = 8m + i$ for some $i \in \{0, 1, 2\}$ and $w_1(BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 8m + 2$, $w_1(CD) = 8m + i$ and $w_1(EF) = 6$, so that P_2 wins by Lemma 4.13.

(c) If $w_1(CD) = 8n + \ell_1$ for some $n < m$ and $\ell_1 \in \{0, 1, \ldots, 7\}$, then the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 2$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins.

(ii) Suppose P_1 removes a positive integer weight from AB and a non-negative integer weight from BC in the first round. Once again, we consider a few subcases:

(a) Suppose $w_1(AB) = 8m+i$ for $i \in \{0,1\}$ and $w_1(BC) = 1$. Then P_2 removes weight $(2-i)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 8m+i,\ w_2(BC) = 1,\ w_2(CD) = 8m+3,\ w_2(EF) = 4+i,$$

which is of the form given by (4.4) with either $r = 1$ (when $i = 1$) or $r = 0$ (when $i = 0$), and by what we have already proved in §7.4 and §8.5, we know that P_1 loses.

(b) Suppose $w_1(AB) = 8m+i$ for $i \in \{0,1\}$ and $w_1(BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 8m+i$, $w_1(CD) = 8m+3$ and $w_1(EF) = 6$, and P_2 wins by Lemma 4.13.

(c) Finally, suppose $w_1(AB) = 8n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,7\}$, and as argued above in (ic), this leads to P_2 winning due to Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = k \leqslant 5$. For $k \in \{4,5\}$, P_2 removes weight $(6-k)$ from the edge AB in the second round, leaving P_1 with

$$w_2(AB) = 8m+(k-4),\ w_2(BC) = 1,\ w_2(CD) = 8m+3,\ w_2(EF) = k,$$

which is of the form given by (4.4) with $r = 1$ if $k = 5$, and of the form given by (4.4) with $r = 0$ if $k = 4$, and by what we have already proved in §7.4 and §8.5, we know that P_1 loses. If $k \in \{2,3\}$, then the configuration after the first round is of the form given by Lemma 4.9, with $k \in \{\ell_1, \ell_2\}$ since $\ell_1 = 2$ and $\ell_2 = 3$, and hence, P_2 wins. If $k = 1$, then P_2 removes the edge BC in the second round, leaving P_1 with a galaxy graph consisting of the edges AB, CD and EF, where $w_2(AB) = 8m+2$, $w_2(CD) = 8m+3$ and $w_2(EF) = 1$, so that P_1 loses by Theorem 2.3. Finally, if $k = 0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes the edge BC in the first round, leaving P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 8m+2$, $w_1(CD) = 8m+3$ and $w_1(EF) = 6$, so that P_2 wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.4) is losing on H_1.

8.7. **Details omitted from the proof of our claim that any configuration that is of the form** (4.7)**, with** $r = 2$**, is losing.** An inductive argument has been employed, in §7.7, in proving that a configuration of the form given by (4.7), with $r = 2$, is losing on H_1. The base case for this induction corresponds to $k = 6$ and $s = 2$, resulting in the configuration: $w_0(AB) = w_0(CD) = 8m+2$, $w_0(BC) = 2$ and $w_0(EF) = 6$.

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round (an analogous situation arises if P_1 removes a positive integer weight from AB and a non-negative integer weight from BC). We consider a few subcases of this case:

(a) Suppose $w_1(CD) = 8m+\ell$ for some $\ell \in \{0,1\}$, and $w_1(BC) = t \in \{1,2\}$. Then P_2 removes weight $6-(t+\ell+2)$ from the edge EF in the second round, so that P_1 is left with

$$w_2(AB) = 8m+\ell,\ w_2(BC) = t,\ w_2(CD) = 8m+2,\ w_2(EF) = t+\ell+2,$$

which is of the form (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$), and by what we have already proved in §7.4 and §8.5, we conclude that P_1 loses.

(b) Suppose $w_1(CD) = 8m+\ell$ for some $\ell \in \{0,1\}$, and $w_1(BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 8m+\ell$, $w_1(CD) = 8m+2$ and $w_1(EF) = 6$, and P_2 wins by Lemma 4.13.

(c) Suppose $w_1(CD) = 8n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,7\}$. The resulting configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 2$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant 5$. If $k \in \{4,5\}$, then P_2 removes weight $6\ \ k)$ from $\ B$ in the second round, leaving P_1 with

$$w_2\ \ B) = 8m + \ k\ \ 4),\ w_2\ BC) = 2,\ w_2\ CD) = 8m+2,\ w_2\ EF) = k,$$

which is of the form given by (4.4) with either $r = 0$ (when $k = 4$) or $r = 1$ (when $k = 5$). When $k = 3$, P_2 removes weight 2 from $\ B$ and weight 1 from BC, leaving P_1 with

$$w_2\ \ B) = 8m,\ w_2\ BC) = 1,\ w_2\ CD) = 8m+2,\ w_2\ EF) = 3,$$

which is of the form given by (4.4) with $r = 0$. By what we have already proved in §7.4 and §8.5, we conclude that P_1 loses in each of these cases. When $k = 2$, the resulting configuration is of the form stated in Lemma 4.9, with $m_1 = m_2 = 2m$ and $\ell_1 = \ell_2 = 2$, so that $k \in \{\ell_1, \ell_2\}$, and hence, P_2 wins by Lemma 4.9. If $k = 1$, then $w_1\ BC) > w_1\ EF)$, so that P_2 wins by Lemma 4.11. Finally, if $k = 0$, P_2 wins by Remark 4.12.

(iii) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of $\ B$ and CD. If $w_1\ BC) = 1$, then P_2 removes weight 5 from EF in the second round, leaving P_1 with $w_2\ \ B) = w_2\ CD) = 8m+2 = 2\ 4m+1)$ and $w_2\ BC) = w_2\ EF) = 1$, which is of the form given by (4.4) with $r = 0$, and by what we have already proved in §7.4, we know that P_1 loses. If $w_1\ BC) = 0$, then P_2 is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1\ \ B) = w_1\ CD) = 8m+2$ and $w_1\ EF) = 6$, and P_2 wins by Lemma 4.13.

This completes the proof of the base case for the inductive argument employed in proving that a configuration of the form (4.7), with $r = 2$, is losing on H_1.

8.8. **Proof that any configuration of the form given by** (4.4) **with** $r = 3$ **is losing.** We employ a sequence of inductive arguments in our proof for establishing that a configuration on H_1 that is of the form given by (4.4) with $r = 3$ is losing. First, we show that any configuration that is of the form given by (4.4) with $r = 3$ and $k = 7$ is losing, i.e. for any $m \in\ \ _0$, we focus on the configuration

$$w_0\ \ B) = w_0\ CD) = 8m+3,\ w_0\ BC) = 1,\ w_0\ EF) = 7 \tag{8.5}$$

and consider the first round of the game played on this configuration:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round (this is analogous to P_1 removing a positive integer weight from $\ B$ and a non-negative integer weight from BC in the first round). The following subcases are possible:

(a) Suppose $w_1\ CD) = 8m+i$ for some $i \in \{0,1,2\}$ and $w_1\ BC) = 1$. Then P_2 removes weight $3\ \ i)$ from the edge EF in the second round, leaving P_1 with

$$w_2\ \ B) = 8m+3,\ w_2\ BC) = 1,\ w_2\ CD) = 8m+i,\ w_2\ EF) = 4+i,$$

which is either of the form given by (4.4) with $r = 0$ (when $i = 0$) or $r = 1$ (when $i = 1$), or of the form given by (4.6) with $r = 2$, and by what we have already proved in §7.4, §8.5 and §7.6, we conclude that P_1 loses.

(b) Suppose $w_1\ CD) = 8m+i$ for some $i \in \{0,1,2\}$, and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1\ \ B) = 8m+3$, $w_1\ CD) = 8m+i$ and $w_1\ EF) = 7$, and P_2 wins by Lemma 4.13.

(c) Finally, suppose $w_1\ CD) = 8n+\ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1,\ \ ,7\}$. The resulting configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 3$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant 6$. If $k \in \{4,5,6\}$, then P_2 removes weight $7\ \ k)$ from CD in the second round, leaving P_1 with

$$w_2\ \ B) = 8m+3,\ w_2\ BC) = 1,\ w_2\ CD) = 8m+\ k\ \ 4),\ w_2\ EF) = k,$$

which is either of the form given by (4.4) with $r = 0$ (when $k = 4$) or $r = 1$ (when $k = 5$), or of the form given by (4.6) with $r = 2$ (when $k = 6$), and by what we have already proved in §7.4, §8.5 and §7.6, we conclude that P_1 loses. If $k \in \{1,2,3\}$, the configuration after the first round is of the form stated in Lemma 4.9 with $\ell_1 = \ell_2 = 3$ and $w_1(EF) = k \leqslant \min\{\ell_1, \ell_2\}$, and P_2 wins by Lemma 4.9. When $k = 0$, P_2 wins by Remark 4.12.

(iii) Suppose P_1 removes the edge BC in the first round, but leaves the edge-weights of AB and CD undisturbed. Then, after the first round, P_2 is left with a galaxy graph, consisting of the edges AB, CD and EF, with $w_1(AB) = w_1(CD) = 8m+3$ and $w_1(EF) = 7$, and P_2 wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.5) is losing.

Suppose, for some $K \in \mathbb{N}$, with $K \geqslant 7$, we have proved that any configuration that is of the form given by (4.4) with $r = 3$ and $w_0(EF) = k \leqslant K$, is losing on H_1. We now consider a configuration of the form

$$w_0(AB) = 2^{R+1}m+3,\ w_2(BC) = K-L-2,\ w_2(CD) = 2^{R+1}m+L,\ w_2(EF) = K+1, \tag{8.6}$$

where $R = f(K+1)$ and $L \in \{3,4,\ldots,K-3\}$. The first round of the game played on this configuration may unfold as follows:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. This can be subdivided into the following cases:

(a) If $w_1(CD) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,\ldots,L-1\}$ and $w_1(BC) = t$ for some $t \in \{1,2,\ldots,K-L-2\}$, P_2 removes weight $(K+1)-(3+t+\ell)$ from EF in the second round. Note that $\ell \leqslant L-1$ and $t \leqslant K-L-2$ together imply $(3+t+\ell) \leqslant K$. This leaves P_1 with

$$w_2(AB) = 2^{f(3+t+\ell)+1}2^{R-f(3+t+\ell)}m+3,\ w_2(BC) = t,$$
$$w_2(CD) = 2^{f(3+t+\ell)+1}2^{R-f(3+t+\ell)}m+\ell,\ w_2(EF) = (3+t+\ell),$$

which is of the same form as (4.4) with $r = 3$ and $w_2(EF) = (3+t+\ell) \leqslant K$. Consequently, P_1 loses by our induction hypothesis.

(b) Suppose $w_1(CD) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,\ldots,L-1\}$, and $w_1(BC) = 0$. This leaves P_2 with a galaxy graph comprising the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+3$, $w_1(CD) = 2^{R+1}m+\ell$ and $w_1(EF) = (K+1)$. Since $L \leqslant (K-3)$ and $\ell \leqslant (L-1)$, P_2 wins by Lemma 4.13.

(c) If $w_1(CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 3$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, in the first round. Once again, we consider the possible cases:

(a) Suppose $w_1(AB) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,2\}$, and $w_1(BC) = t \in \{1,2,\ldots,K-L-2\}$. If (i) $\ell \in \{0,1\}$, (ii) or if $\ell = 2$ and $t \geqslant 2$, (iii) or if $\ell = 2$, $t = 1$ and $L \equiv i \bmod 4$ for some $i \in \{0,1,3\}$, then P_2 removes weight $(K+1)-(\ell+t+L)$ from EF in the second round. Note that $\ell \leqslant 2$ and $t \leqslant K-L-2$ together imply $(\ell+t+L) \leqslant K$. This leaves P_1 with

$$w_2(AB) = 2^{f(\ell+t+L)+1}2^{R-f(\ell+t+L)}m+\ell,\ w_2(BC) = t,$$
$$w_2(CD) = 2^{f(\ell+t+L)+1}2^{R-f(\ell+t+L)}m+L,\ w_2(EF) = (\ell+t+L),$$

which is of the form given by (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$), or of the form given by (4.7) with $r = 2$ when $\ell = 2$ and $t \geqslant 2$, or of the form given by (4.6) wth $r = 2$ when $\ell = 2$, $t = 1$ and $L \equiv i \bmod 4$ for some $i \in \{0,1,3\}$. By what we have already proved in §7.4, §8.5, §7.6 and §7.7, we conclude that in each of these cases, P_1 loses. If $\ell = 2$,

$t = 1$ and $L \equiv 2 \bmod 4$, P_2 removes weight $\ K+1) \quad L \quad 1)$ from EF in the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{f\ L \quad 1)+1} 2^{R \quad f\ L \quad 1)} m + 2,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L \quad 1)+1} 2^{R \quad f\ L \quad 1)} m + L,\ w_2\ EF) = \ L \quad 1),$$

which is of the form given by (4.5) with $r = 2$, and by what we have already proved in §7.6, we conclude that P_1 loses.

(b) Suppose $w_1 \quad B) = 2^{R+1}m + \ell$ for some $\ell \in \{0,1,2\}$, and $w_1\ BC) = 0$. This leaves P_2 with a galaxy graph, consisting of the edges $\ B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m + \ell$, $w_1\ CD) = 2^{R+1}m + L$ and $w_1\ EF) = \ K+1)$. Since $L \leqslant \ K \quad 3)$ and $\ell \leqslant 2$, P_2 wins by Lemma 4.13.

(c) If $w_1 \quad B) = 2^{R+1}n + \ell_1$ for some $n < m$ and some $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = L$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$. For $k \in \{K \quad L+1, K \quad L+2, \quad ,K\}$, P_2 removes weight $\ K+1 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ k)+1} 2^{R \quad f\ k)} m + 3,\ w_2\ BC) = \ K \quad L \quad 2),$$
$$w_2\ CD) = 2^{f\ k)+1} 2^{R \quad f\ k)} m + \{k + L \quad K+1)\},\ w_2\ EF) = k,$$

which is of the form given by (4.4) with $r = 3$ and $w_2\ EF) \leqslant K$ if $k \geqslant K \quad L+4$, of the form given by (4.7) with $r = 2$ when $k = K \quad L+3$ and $L < K \quad 3$, of the form given by (4.6) with $r = 2$ when $k = K \quad L+3$ and $L = K \quad 3$, of the form given by (4.4) with $r = 1$ when $k = K \quad L+2$, and of the form given by (4.4) with $r = 0$ when $k = K \quad L+1$. In the first of these cases, P_1 loses by our induction hypothesis, and in the remaining cases, P_1 loses by what we have already established in §7.7, §7.6, §8.5 and §7.4 respectively.

For $k \in \{4,5, \quad ,K \quad L\}$, the configuration at the end of the first round is of the same form as that mentioned in Lemma 4.11, with $\ell_1 = 3$, $\ell_2 = L$, so that $w_2\ EF) = k > 3 = \min\{\ell_1, \ell_2\}$ and $k \quad \min\{\ell_1,\ell_2\} \leqslant K \quad L \quad 3 < K \quad L \quad 2 = w_2\ BC)$. Consequently, P_2 wins by Lemma 4.11. For $k \in \{1,2,3\}$, the configuration at the end of the first round is of the same form as that mentioned in Lemma 4.9, with $\ell_1 = 3$, $\ell_2 = L$, so that $\min\{\ell_1,\ell_2\} \geqslant k = w_1\ EF)$, and P_2 wins by Lemma 4.9. If $k = 0$, P_2 wins by Remark 4.12.

(iv) Finally, suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of $\ B$ and CD. Then $w_1\ BC) = t \in \{0,1, \quad ,K \quad L \quad 3\}$. For $t \geqslant 1$, P_2 simply removes weight $\ K+1) \quad 3+t+L)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ 3+t+L)+1} 2^{R \quad f\ 3+t+L)} m + 3,\ w_2\ BC) = t,$$
$$w_2\ CD) = 2^{f\ 3+t+L)+1} 2^{R \quad f\ 3+t+L)} m + L,\ w_2\ EF) = \ 3+t+L),$$

which is of the same form as (4.4) with $r = 3$ and $w_2\ EF) = \ 3+t+L) \leqslant K$, so that P_1 loses by our induction hypothesis. If $t = 0$, then, at the end of the first round, P_2 is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m + 3$, $w_1\ CD) = 2^{R+1}m + L$ and $w_1\ EF) = \ K+1)$. Since $L \leqslant \ K \quad 3)$, it is evident from Lemma 4.13 that P_2 wins.

This completes the proof of our claim that the configuration in (8.6) is losing on H_1, and also concludes our inductive proof of our claim that any configuration that is of the form given by (4.4), with $r = 3$, is losing.

8.9. **Proof that a configuration that is either of the form given by** (4.5) **or of the form given by** (4.6) **is losing for** $r = 4$ **and** $s = 1$**.** Our argument is, as above, inductive. The base case corresponding to (4.6) with

$r=4$ and $s=1$ is obtained by setting $k=12$, which yields the configuration

$$w_0(AB)=16m+4,\ w_0(BC)=1,\ w_0(CD)=16m+7,\ w_0(EF)=12 \tag{8.7}$$

We consider the first round of the game played on this configuration:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. This case can be subdivided into the following scenarios:

(a) Suppose $w_1(CD)=16m+\ell$ for some $\ell\in\{0,1,\ldots,6\}$, and $w_1(BC)=1$. If $\ell\in\{0,1,2,3\}$, then P_2 removes weight $(7-\ell)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB)=16m+4,\ w_2(BC)=1,\ w_2(CD)=16m+\ell,\ w_2(EF)=(5+\ell),$$

which is of the form given by (4.4) with $r=0$ when $\ell=0$, of the form given by (4.4) with $r=1$ when $\ell=1$, of the form given by (4.6) with $r=2$ when $\ell=2$, and of the form given by (4.4) with $r=3$ when $\ell=3$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we conclude that P_1 loses in each of these cases. On the other hand, if $\ell\in\{4,5,6\}$, P_2 removes weight $(15-\ell)$ from the edge EF in the second round, leaving P_1 with the configuration

$$w_2(AB)=16m+4=4(4m+1),\ w_2(BC)=1,$$
$$w_2(CD)=16m+\ell=4(4m+1)+(\ell-4),\ w_2(EF)=(\ell-3),$$

which is of the form given by (4.4) with $r=0$, and by what we have already proved in §7.4, we conclude that P_1 loses.

(b) If $w_1(CD)=16m+\ell$ for some $\ell\in\{0,1,\ldots,6\}$, and $w_1(BC)=0$, then P_2, after the first round, is left with a galaxy graph, comprising the edges AB, CD and EF, such that $w_1(AB)=16m+4$, $w_1(CD)=16m+\ell$ and $w_1(EF)=12$, and it is evident from Lemma 4.13 that P_2 wins.

(c) If $w_1(CD)=16n+\ell_1$ for some $n<m$ and some $\ell_1\in\{0,1,\ldots,15\}$, the configuration, after the first round, is of the form given by Lemma 4.9 with $m_1=n$, $m_2=m$ and $\ell_2=4$. Since $n<m \implies m_1\neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, in the first round. Once again, we consider all possible subcases:

(a) Suppose $w_1(AB)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$, and $w_1(BC)=1$. Then P_2 removes weight $(4-\ell)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB)=16m+\ell,\ w_2(BC)=1,\ w_2(CD)=16m+7,\ w_2(EF)=8+\ell,$$

which is of the form given by (4.4) with $r=0$ when $\ell=0$, of the form given by (4.4) with $r=1$ when $\ell=1$, of the form given by (4.6) with $r=2$ when $\ell=2$, and of the form given by (4.4) with $r=3$ when $\ell=3$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we know that P_1 loses.

(b) If $w_1(AB)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$, and $w_1(BC)=0$, P_2 is left, at the end of the first round, with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB)=16m+\ell$, $w_1(CD)=16m+7$ and $w_1(EF)=12$, and P_2 wins by Lemma 4.13.

(c) If $w_1(AB)=16n+\ell_1$ for some $n<m$ and some $\ell_1\in\{0,1,\ldots,15\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1=n$, $m_2=m$ and $\ell_2=7$. Since $n<m \implies m_1\neq m_2$, hence P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF)=k\leqslant 11$. If $k\in\{8,9,10,11\}$, P_2 removes weight $(12-k)$ from AB in the second round, so that P_1 is left with

$$w_2(AB)=16m+(k-8),\ w_2(BC)=1,\ w_2(CD)=16m+7,\ w_2(EF)=k,$$

which is of the form given by (4.4) with $r=0$ when $k=8$, of the form given by (4.4) with $r=1$ when $k=9$, of the form given by (4.6) with $r=2$ when $k=10$, and of the form given by (4.4) with

$r=3$ when $k=11$, and by what we have already proved in §7.4, §8.5, §7.6 and §8.8, we know that P_1 loses. If $k \in \{1,2,3,4,7\}$, the configuration at the end of the first round is of the same form as described in Lemma 4.9, with $\ell_1 = 4$ and $\ell_2 = 7$, so that we either have $\min\{\ell_1,\ell_2\} \geqslant k$, or we have $k \in \{\ell_1,\ell_2\}$, and in each of these cases, P_2 wins by Lemma 4.9. If $k=0$, P_2 wins by Remark 4.12.

If $k \in \{5,6\}$, P_2 removes weight $(12-k)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 16m+4,\ w_2(BC) = 1,\ w_2(CD) = 16m+(k-5),\ w_2(EF) = k,$$

which is of the form given by (4.4) with either $r=0$ (when $k=5$) or $r=1$ (when $k=6$), and by what we have already proved in §7.4 and §8.5, we know that P_1 loses.

(iv) Finally, suppose P_1 removes the edge BC, without disturbing the edge-weights of AB and CD, in the first round. This leaves P_2 with a galaxy graph, consisting of the edges AB, CD and EF, with $w_2(AB) = 16m+4$, $w_2(CD) = 16m+7$ and $w_2(EF) = 12$, and P_2 wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.7) is losing on H_1.

Next, setting $r=4$, $s=1$ and one of $k=1$, $k=2$ and $k=3$ in (4.5), we obtain, respectively:

$$w_0(AB) = w_0(CD) = 2m+4 = 2(m+2),\ w_0(BC) = w_0(EF) = 1,$$
$$w_0(AB) = 4m+4 = 4(m+1),\ w_0(BC) = 1,\ w_0(CD) = 4m+5 = 4(m+1)+1,\ w_0(EF) = 2,$$
$$w_0(AB) = 4m+4 = 4(m+1),\ w_0(BC) = 1,\ w_0(CD) = 4m+6 = 4(m+1)+2,\ w_0(EF) = 3,$$

each of which is losing since it is of the form given by (4.4) with $r=0$. This establishes the base case for the inductive proof of our claim that a configuration of the form given by (4.5) with $r=4$ and $s=1$ is losing.

We now come to the inductive steps of our proof. Suppose, for some $K \in \mathbb{N}$, we have shown that any configuration on H_1 that is either of the form (4.5) or of the form (4.6), with $r=4$ and $s=1$, and satisfying the inequality $k \leqslant K$, is losing.

8.9.1. *When $K \geqslant 8$ and $K \equiv i \bmod 8$ for some $i \in \{0,1,2\}$.* For any $m \in \mathbb{N}_0$, we focus on the configuration

$$w_0(AB) = 2^{R+1}m+4,\ w_0(BC) = 1,\ w_0(CD) = 2^{R+1}m+(K+4),\ w_0(EF) = (K+1), \tag{8.8}$$

where $R = f(K+1)$ – this is of the form given by (4.5) with $r=4$ and $s=1$. We consider the first round of the game played on this configuration:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. We subdivide this into the following cases:

(a) If $w_1(CD) = 2^{R+1}m+(K+1)$, then, irrespective of what the value of $w_1(BC)$ is, the configuration after the first round is of the form stated in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = (K+1)$ and $w_1(EF) = (K+1) \in \{\ell_1,\ell_2\}$, so that P_2 wins by Lemma 4.9.

(b) Suppose $w_1(CD) = 2^{R+1}m+L$ where $L \in \{7,8,\ldots,K-5\}$, $L \equiv j \bmod 8$ for some $j \in \{0,1,2,3,7\}$, and $w_1(BC) = 1$. Here, P_2 removes weight $(K+1)-(L+5)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+5)+1}2^{R-f(L+5)}m+4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(L+5)+1}2^{R-f(L+5)}m+L,\ w_2(EF) = L+5,$$

where $(L+5) \geqslant 12$ and $(L+5) \equiv j' \bmod 8$ for some $j' \in \{0,4,5,6,7\}$. Therefore, this configuration is of the form given by (4.6) with $r=4$, $s=1$ and $w_2(EF) = (L+5) \leqslant K$, so that P_1 loses by our induction hypothesis.

(c) Suppose $w_1(CD) = 2^{R+1}m+L$ where $L \in \{1,2,\ldots,K+3\}$, $L \equiv j \bmod 8$ for some $j \in \{4,5,6\}$, and $w_1(BC) = 1$. Recall that $K \equiv i \bmod 8$, for some $i \in \{0,1,2\}$, for the entirety of §8.9.1. The following cases are, consequently, taken care of under this scenario:

(i) $i=0$ and $L \in \{K-4,K-3,K-2\}$,

(ii) $i = 1$ and $L \in \{K \quad 4, K \quad 3, K+3\}$,
(iii) $i = 2$ and $L \in \{K \quad 4, K+2, K+3\}$.

Here, P_2 removes weight $\ K+1) \quad\ L \quad 3)$ from the edge EF in the second round. Note that since $L \leqslant\ K+3)$, we have $\ L \quad 3) \leqslant K$. The configuration after the second round is

$$w_2 \quad B) = 2^{f\ L\ \ 3)+1} 2^{R\ \ f\ L\ \ 3)} m + 4,\ w_2\ BC) = 1,$$

$$w_2\ CD) = 2^{f\ L\ \ 3)+1} 2^{R\ \ f\ L\ \ 3)} m + L,\ w_2\ EF) = L \quad 3,$$

and as $\ L \quad 3) \equiv j' \bmod 8$ for some $j' \in \{1,2,3\}$, this is of the form given by (4.5) with $r = 4$ and $s = 1$. Since $w_2\ EF) =\ L \quad 3) \leqslant K$, P_1 loses by our induction hypothesis.

(d) Suppose $w_1\ CD) = 2^{R+1}m + L$ where $L \in \{K \quad 1, K\}$, and $w_1\ BC) = 1$. Then P_2 removes weight $\ L \quad K+4)$ from the edge $\quad B$ in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m +\ K \quad L),\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m + L,\ w_2\ EF) =\ K+1),$$

which is of the form given by (4.4) with either $r = 0$ (when $L = K$) or $r = 1$ (when $L = K \quad 1$), and by what we have already proved in §7.4 and §8.5, we know that P_1 loses. Likewise, when $K \equiv i \bmod 8$ for $i \in \{1,2\}$, $w_1\ CD) = 2^{R+1}m +\ K \quad 2)$ and $w_1\ BC) = 1$, P_2 removes weight 2 from $\quad B$ in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 2,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m +\ K \quad 2),\ w_2\ EF) =\ K+1),$$

which is of the form given by (4.6) with $r = 2$ and $s = 1$, and by what we have already proved in §7.6, we conclude that P_1 loses. When $K \equiv 2 \bmod 8$, $w_1\ CD) = 2^{R+1}m +\ K \quad 3)$ and $w_1\ BC) = 1$, P_2 removes weight 1 from $\quad B$, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 3,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m +\ K \quad 3),\ w_2\ EF) =\ K+1),$$

which is of the same form as (4.4) with $r = 3$, and by what we have already proved in §8.8, we know that P_1 loses. If $K \equiv i \bmod 8$ for $i \in \{0,1\}$, $w_1\ CD) = 2^{R+1}m +\ K+2)$, and $w_1\ BC) = 1$,

(i) P_2 removes BC and weight 1 from $\quad B$ in the second round when $K \equiv 0 \bmod 8$,
(ii) P_2 removes BC and weight 3 from $\quad B$ in the second round when $K \equiv 1 \bmod 8$.

Consequently, P_1 is left with a galaxy graph, consisting of the edges $\quad B$, CD and EF, with

$$w_2 \quad B) = 2^{R+1}m + 3,\ w_2\ CD) = 2^{R+1}m +\ 8n+2),\ w_2\ EF) =\ 8n+1)$$

when $K = 8n$ for some $n \in \quad$, or

$$w_2 \quad B) = 2^{R+1}m + 1,\ w_2\ CD) = 2^{R+1}m +\ 8n+3),\ w_2\ EF) =\ 8n+2)$$

when $K =\ 8n+1)$ for some $n \in \quad$. In each case, the triple is balanced, and P_1 loses by Theorem 2.3. Likewise, when $K \equiv 0 \bmod 8$, $w_1\ CD) = 2^{R+1}m +\ K+3)$ and $w_1\ BC) = 1$, P_2 removes BC and weight 2 from $\quad B$ in the second round, leaving P_1 with a galaxy graph consisting of the edges $\quad B$, CD and EF, where (once again, writing $K = 8n$ for some $n \in \quad$)

$$w_2 \quad B) = 2^{R+1}m + 2,\ w_2\ CD) = 2^{R+1}m +\ 8n+3),\ w_2\ EF) =\ 8n+1),$$

and as this triple is balanced, P_1 loses by Theorem 2.3.

(e) Suppose $w_1\ CD) = 2^{R+1}m + L$ for some $L \in \{0, 1, \quad, 6\}$, and $w_1\ BC) = 1$. If $L \in \{0,1,2,3\}$, then P_2 removes weight $\ K+1) \quad\ L+5)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m + L,\ w_2\ EF) =\ L+5),$$

which is of the form given by (4.4) with $r = 0$ when $L = 0$, of the form given by (4.4) with $r = 1$ when $L = 1$, of the form given by (4.6) with $r = 2$ when $L = 2$, and of the form given by (4.4) with $r = 3$ when $L = 3$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we conclude that P_1 loses. If $L \in \{4,5,6\}$, P_2 removes weight $(K+1)-(L-3)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+4 = 4\left(2^{R-1}m+1\right),\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{R+1}m+L = 4\left(2^{R-1}m+1\right)+(L-4),\ w_2(EF) = (L-3),$$

which is of the form given by (4.4) with $r=0$, and by what we have already proved in §7.4, we know that P_1 loses.

(f) If $w_1(CD) = 2^{R+1}m+L$ for some $L \in \{0,1,\ldots,K-4\}$ and $w_1(BC) = 0$, then P_2, after the first round, is left with a galaxy graph comprising the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+L$ and $w_1(EF) = (K+1)$, and P_2 wins by Lemma 4.13.
Since $K \equiv i \bmod 8$ for some $i \in \{0,1,2\}$, we write $K = 8(n+1)+i$ for some $n \in \mathbb{N}_0$. When $w_1(CD) = 2^{R+1}m+L$ for $L = (K-3) = (8n+5+i)$, and $w_1(BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+(8n+5+i)$ and $w_1(EF) = (8n+9+i)$. Since the triple $\left(2^{R+1}m+4, 2^{R+1}m+(8n+5+i), (8n+i+1)\right)$ is balanced, the triple $\left(2^{R+1}m+4, 2^{R+1}m+(8n+5+i), (8n+9+i)\right)$ is not, and P_2 wins by Theorem 2.3. Likewise, when $w_1(CD) = 2^{R+1}m+L$ and $w_1(BC) = 0$

(i) with $L = (K-2)$ and $K \equiv i \bmod 8$ for $i \in \{0,1\}$,
(ii) or with $L = (K-1)$ and $K \equiv 0 \bmod 8$,
(iii) or with $L = (K+2)$ and $K \equiv 2 \bmod 8$,
(iv) or with $L = (K+3)$ and $K \equiv i \bmod 8$ for $i \in \{1,2\}$,

the triple $\left(2^{R+1}m+4, 2^{R+1}m+L, (L-4)\right)$ is balanced, and hence, the triple $\left(2^{R+1}m+4, 2^{R+1}m+L, (K+1)\right)$ not, allowing P_2 to win by Theorem 2.3.
When $w_1(CD) = 2^{R+1}m+L$ with $L = K$ and $w_1(BC) = 0$, P_2, after the first round, is left with a galaxy graph comprising the edges AB, CD and EF, with $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+K$ and $w_1(EF) = (K+1)$. Since the triple $\left(2^{R+1}m+1, 2^{R+1}m+K, K+1\right)$ is balanced when $K \equiv i \bmod 8$ for $i \in \{0,2\}$, and the triple $\left(2^{R+1}m+3, 2^{R+1}m+K, K+1\right)$ is balanced when $K \equiv 1 \bmod 8$, we conclude that P_2 wins by Theorem 2.3. Very similar arguments work when $w_1(CD) = 2^{R+1}m+L$ and $w_1(BC) = 0$ with

(i) $L = (K-2)$ and $K \equiv 2 \bmod 8$,
(ii) $L = (K-1)$ and $K \equiv i \bmod 8$ for $i \in \{1,2\}$,
(iii) $L = (K+2)$ and $K \equiv i \bmod 8$ for $i \in \{0,1\}$,
(iv) $L = (K+3)$ and $K \equiv 0 \bmod 8$.

(g) If $w_1(CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, in the first round. We divide the analysis into the following cases:

(a) Suppose $w_1(AB) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,2,3\}$. Writing $(K+1) = (a_R a_{R-1} \ldots a_1 a_0)_2$, we see that $a_2 = 0$ since $K \equiv i \bmod 8$ for some $i \in \{0,1,2\}$. Moreover, writing $(K+4) = (c_R c_{R-1} \ldots c_1 c_0)_2$, we see that $c_i = a_i$ for all $i \geqslant 3$, and $c_2 = 1$. On the other hand, writing $\ell = (b_R b_{R-1} \ldots b_1 b_0)_2$, we have $b_i = 0$ for all $i \geqslant 2$. Consequently, P_2 wins by the second criterion stated in Theorem 4.8.

(b) If $w_1(AB) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = (K+4)$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = k \leqslant K$. If $k \in \{12, 13, \ldots, K\}$ and $k \equiv j \bmod 8$ for some $j \in \{0,4,5,6,7\}$, then P_2 removes weight $(K+4) - (k-5)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m + 4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m + (k-5),\ w_2(EF) = k,$$

which is of the form given by (4.6) with $r = 4$, $s = 1$ and $w_2(EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. If $k \in \{1, 2, \ldots, K\}$ and $k \equiv j \bmod 8$ for some $j \in \{1,2,3\}$, then P_2 removes weight $(K+4) - (k+3)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m + 4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m + (k+3),\ w_2(EF) = k,$$

which is of the form given by (4.5) with $r = 4$, $s = 1$ and $w_2(EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. Note that this last case covers $k \in \{1,2,3,9,10,11\}$. If $k = 4$, the configuration after the first round is of the form stated in Lemma 4.11 with $\ell_1 = 4$ and $\ell_2 = (K+4)$, so that $k \in \{\ell_1, \ell_2\}$ and P_2 wins by Lemma 4.11. If $k \in \{5,6,7,8\}$, P_2 removes weight $(K+4) - (k-5)$ from the edge CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m + 4,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + (k-5),\ w_2(EF) = k,$$

which is of the same form as (4.4) with $r = 0$ when $k = 5$, of the form given by (4.4) with $r = 1$ when $k = 6$, of the form given by (4.6) with $r = 2$ when $k = 7$, and of the form given by (4.4) with $r = 3$ when $k = 8$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we know that P_1 loses. If $k = 0$, P_2 wins by Remark 4.12.

(iv) Finally, suppose P_1 removes the edge BC in the first round, without disturbing the edge-weights of AB and CD. This leaves P_2 with a galaxy graph consisting of the edges AB, CD and EF, where $w_2(AB) = 2^{R+1}m + 4$, $w_2(CD) = 2^{R+1}m + (K+4)$ and $w_2(EF) = (K+1)$. Writing $K = 8n + i$ for some $n \in \mathbb{N}$ and $i \in \{0,1,2\}$, we see that the triple $\left(2^{R+1}m + 4, 2^{R+1}m + 8n + 4 + i, 8n + i\right)$ is balanaced, so that P_2 wins by Theorem 2.3.

This completes the proof of our claim that the configuration in (8.8) is losing on H_1.

8.9.2. *When $K \geqslant 12$ and $K \equiv i \bmod 8$ for some $i \in \{3,4,5,6,7\}$.* For $m \in \mathbb{N}_0$, we focus on the configuration

$$w_0(AB) = 2^{R+1}m + 4,\ w_0(BC) = 1,\ w_0(CD) = 2^{R+1}m + (K-4),\ w_0(EF) = (K+1), \tag{8.9}$$

where $R = f(K+1)$. The first round of the game played on this configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. The following possibilities can arise:

(a) Let $w_1(CD) = 2^{R+1}m + L$ for some $L \in \{0, 1, \ldots, K-5\}$, with $L \equiv j \bmod 8$ for some $j \in \{4,5,6\}$ and $w_1(BC) = 1$. In this case, P_2 removes weight $(K+1) - (L-3)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-3)+1}2^{R-f(L-3)}m + 4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(L-3)+1}2^{R-f(L-3)}m + L,\ w_2(EF) = (L-3),$$

which is of the form given by (4.5) with $r = 4$, $s = 1$ and $w_2(EF) = (L-3) \leqslant (K-8) < K$, so that P_1 loses by our induction hypothesis.

(b) Let $w_1\ CD) = 2^{R+1}m+L$ for some $L \in \{7,8, \quad ,K \quad 5\}$, with $L \equiv j \bmod 8$ for some $j \in \{0,1,2,3,7\}$, and $w_1\ BC) = 1$. In this case, P_2 removes weight $\ K+1) \quad L+5)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+5)+1}2^{R \quad f\ L+5)}m+4,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L+5)+1}2^{R \quad f\ L+5)}m+L,\ w_2\ EF) = \ L+5),$$

which is of the form given by (4.6) with $r = 4$, $s = 1$ and $w_2\ EF) = \ L+5) \leqslant K$, so that P_1 loses by our induction hypothesis.

(c) Let $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{0,1,2,3\}$, and $w_1\ BC) = 1$. In this case, P_2 removes weight $\ K+1) \quad L+5)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m+L,\ w_2\ EF) = \ L+5),$$

which is of the form given by (4.4) with $r = 0$ if $L = 0$, of the form given by (4.4) with $r = 1$ if $L = 1$, of the form given by (4.6) with $r = 2$ if $L = 2$, and of the form (4.4) with $r = 3$ when $L = 3$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we know that P_1 loses in each of these cases.

(d) If $w_1\ CD) = 2^{R+1}m+L$ for some $L \in \{0,1, \quad ,K \quad 5\}$ and $w_1\ BC) = 0$, the configuration at the end of the first round is that on a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1\ CD) = 2^{R+1}m+L$ and $w_1\ EF) = \ K+1)$, and P_2 wins by Lemma 4.13.

(e) Finally, if $w_1\ CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the resulting configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\ B$, and a non-negative integer weight from BC, in the first round. We consider the possible subcases:

(a) Suppose $w_1 \quad B) = 2^{R+1}m+L$ for some $L \in \{0,1,2,3\}$ and $w_1\ BC) = 1$. If $L \in \{0,1,3\}$, or if $L = 2$ and $K \equiv i \bmod 8$ for some $i \in \{3,4,5,7\}$, P_2 removes weight $\ 4 \quad L)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+L,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m+\ K \quad 4),\ w_2\ EF) = \ K+L \quad 3),$$

which is of the form (4.4) with $r = 0$ when $L = 0$, of the form (4.4) with $r = 1$ when $L = 1$, of the form (4.6) with $r = 2$ when $L = 2$, and of the form (4.4) with $r = 3$ when $L = 3$. By what we have already proved in §7.4, §8.5, §7.6 and §8.8, we know that P_1 loses. If $L = 2$ and $K \equiv 6 \bmod 7$, then P_2 removes weight 6 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+2,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m+\ K \quad 4),\ w_2\ EF) = \ K \quad 5),$$

which is of the form (4.5) with $r = 2$, and by what we have already proved in §7.6, we conclude that P_1 loses.

(b) If $w_1 \quad B) = 2^{R+1}m+L$ for some $L \in \{0,1,2,3\}$ and $w_1\ BC) = 0$, P_2 is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m+L$, $w_1\ CD) = 2^{R+1}m+\ K \quad 4)$ and $w_1\ EF) = \ K+1)$ and she wins by Lemma 4.13.

(c) Finally, if $w_1 \quad B) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the resulting configuration, after the first round, is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = \ K \quad 4)$. Since $n < m \implies m_1 \neq m_2$, hence P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$. To begin with, note that if $k \in \{0,1,2,3,4,\ K \quad 4)\}$, then the configuration after the first round is of the form stated in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = \ \ K \quad 4)$ and $w_2 \ \ EF) = k$ either satisfying $\min\{\ell_1, \ell_2\} \geqslant k$ or satisfying $k \in \{\ell_1, \ell_2\}$. Consequently, P_2 wins by Lemma 4.9.

If $k \leqslant K \quad 8$ and $k \equiv j \bmod 8$ for some $j \in \{1,2,3\}$, then P_2 removes weight $\ \ K \quad 4) \qquad k+3)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \ \ k)+1} 2^{R \quad f \ \ k)} m + 4, \ w_2 \ \ BC) = 1,$$
$$w_2 \ \ CD) = 2^{f \ \ k)+1} 2^{R \quad f \ \ k)} m + \ \ k+3), \ w_2 \ \ EF) = k,$$

which is of the form (4.5) with $r = 4$, $s = 1$ and $w_2 \ \ EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. Note that this case covers $k \in \{9, 10, 11\}$. If $k \geqslant 12$ and $k \equiv j \bmod 8$ for some $j \in \{0,4,5,6,7\}$, P_2 removes weight $\ \ K \quad 4) \qquad k \quad 5)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \ \ k)+1} 2^{R \quad f \ \ k)} m + 4, \ w_2 \ \ BC) = 1,$$
$$w_2 \ \ CD) = 2^{f \ \ k)+1} 2^{R \quad f \ \ k)} m + \ \ k \quad 5), \ w_2 \ \ EF) = k,$$

which is of the form (4.6) with $r = 4$, $s = 1$ and $w_2 \ \ EF) = k \leqslant K$, and hence, P_1 loses by our induction hypothesis. Note that this case covers the following:

(a) when $K \equiv 3 \bmod 8$ and $k \in \{K \quad 7, K \quad 6, K \quad 5, K \quad 3\}$,
(b) when $K \equiv 4 \bmod 8$ and $k \in \{K \quad 7, K \quad 6, K \quad 5, K\}$,
(c) when $K \equiv 5 \bmod 8$ and $k \in \{K \quad 7, K \quad 6, K \quad 5, K \quad 1, K\}$,
(d) when $K \equiv 6 \bmod 8$ and $k \in \{K \quad 7, K \quad 6, K \quad 2, K \quad 1, K\}$,
(e) when $K \equiv 7 \bmod 8$ and $k \in \{K \quad 7, K \quad 3, K \quad 2, K \quad 1, K\}$.

If $k = \ \ K \quad 6)$ and $K \equiv 7 \bmod 8$, we write $K = \ \ 8n+7)$ for some $n \in \quad$, so that $w_1 \quad B) = 2^{R+1}m + 4$, $w_1 \ \ CD) = 2^{R+1}m + \ \ 8n+3)$ and $w_1 \ \ EF) = \ \ 8n+1)$, and P_2 wins by Theorem 4.8. If $k = \ \ K \quad 5)$ and $K \equiv 6 \bmod 8$, we write $K = \ \ 8n+6)$ for some $n \in \quad$, so that $w_1 \quad B) = 2^{R+1}m + 4$, $w_1 \ \ CD) = 2^{R+1}m + \ \ 8n+2)$ and $w_1 \ \ EF) = \ \ 8n+1)$. If $k = \ \ K \quad 5)$ and $K \equiv 7 \bmod 8$, writing $K = \ \ 8n+7)$ for some $n \in \quad$, we have $w_1 \quad B) = 2^{R+1}m + 4$, $w_1 \ \ CD) = 2^{R+1}m + \ \ 8n+3)$ and $w_1 \ \ EF) = \ \ 8n+2)$. In each of these cases, P_2 wins by Theorem 4.8. If $k \in \{K \quad 3, K \quad 2, K \quad 1, K\}$, P_2 removes weight $\ \ K+1 \quad k)$ from the edge $\quad B$ in the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{R+1}m + k + 3 \quad K, \ w_2 \ \ BC) = 1, \ w_2 \ \ CD) = 2^{R+1}m + \ \ K \quad 4), \ w_2 \ \ EF) = k,$$

which is of the form (4.4) with either $r = 0$ (when $k = K \quad 3$) or $r = 1$ (when $k = K \quad 2$) or $r = 3$ (when $k = K$), or of the form (4.6) with $r = 2$ (when $k = K \quad 2$). By what we have already proved in §7.4, §8.5, §8.8 and §7.6, we conclude that P_1 loses.

If $k \in \{5,6,7,8\}$, P_2 removes weight $\ \ K+1 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4, \ w_2 \ \ BC) = 1, \ w_2 \ \ CD) = 2^{R+1}m + \ \ k \quad 5), \ w_2 \ \ EF) = k,$$

which is of the form (4.4) with either $r = 0$ (when $k = 5$) or $r = 1$ (when $k = 6$) or $r = 3$ (when $k = 8$), or of the form (4.6) with $r = 2$ (when $k = 7$). By what we have already proved in §7.4, §8.5, §8.8 and §7.6, we conclude that P_1 loses. Finally, if $k = 0$, P_1 loses by Remark 4.12.

(iv) Finally, if P_1 removes the edge BC in the first round, without disturbing the edge-weights of $\quad B$ and CD, P_2 is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, with $w_1 \quad B) = 2^{R+1}m + 4$, $w_1 \ \ CD) = 2^{R+1}m + \ \ K \quad 4)$ and $w_1 \ \ EF) = \ \ K+1)$, and P_2 wins by Lemma 4.13.

This completes the proof of our claim that (8.9) is losing on H_1. Together with our proof that the configuration in (8.8) is losing on H_1, this completes our inductive proof of the claim that any configuration on H_1 that is either of the form given by (4.5) or of the form given by (4.6) with $r = 4$ and $s = 1$ is losing.

8.10. **Proof that a configuration on H_1 satisfying either (4.5) or (4.6), with $r=4$ and $s=2$, is losing.** The base case corresponding to (4.6), with $r=4$ and $s=2$, is obtained by setting $k=12$, yielding

$$w_0(B)=16m+4,\ w_0(BC)=2,\ w_0(CD)=16m+6,\ w_0(EF)=12 \tag{8.10}$$

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round of the game played on the initial configuration in (8.10):

(a) If $w_1(CD)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$ and $w_1(BC)=t\in\{1,2\}$, P_2 removes weight $(8-\ell-t)$ from the edge EF in the second round, leaving P_1 with

$$w_2(B)=16m+4,\ w_2(BC)=t,\ w_2(CD)=16m+\ell,\ w_2(EF)=\ell+t+4,$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.6) with $r=2$ and $s=1$ (when $\ell=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s=2$ (when $\ell=2$ and $t=2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

(b) If $w_1(CD)=2^{R+1}m+\ell$ for some $\ell\in\{4,5\}$ and $w_1(BC)=t\in\{1,2\}$, then P_2 removes weight $(16-\ell-t)$ from the edge EF in the second round, leaving P_1 with

$$w_2(B)=4(4m+1),\ w_2(BC)=t,\ w_2(CD)=4(4m+1)+\ell-4,\ w_2(EF)=t+\ell-4,$$

which is of the same form as (4.4) with $r=0$, and by what we have already proved in §7.4, we know that P_1 loses.

(c) If $w_1(CD)=16m+\ell$ for some $\ell\in\{0,1,\ldots,5\}$ and $w_1(BC)=0$, P_2 is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1(B)=16m+4$, $w_1(CD)=16m+\ell$ and $w_1(EF)=12$, and P_2 wins by Lemma 4.13.

(d) Finally, if $w_1(CD)=16n+\ell_1$ for some $n<m$ and $\ell_1\in\{0,1,\ldots,15\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1=n$, $m_2=m$ and $\ell_2=4$. Since $n<m \implies m_1\neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from B and a non-negative integer weight from BC in the first round. The following subcases are possible:

(a) Suppose $w_1(B)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$ and $w_1(BC)=t\in\{1,2\}$. If $\ell\in\{0,1,3\}$, or if $\ell=2$ and $t=2$, P_2 removes $(6-\ell-t)$ from EF in the second round, leaving P_1 with

$$w_2(B)=16m+\ell,\ w_2(BC)=t,\ w_2(CD)=16m+6,\ w_2(EF)=\ell+t+6,$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.7) with $r=2$ and $s=2$ (when $\ell=2$ and $t=2$). If $\ell=2$ and $t=1$, P_2 removes weight 7 from the edge EF in the first round, leaving P_1 with

$$w_2(B)=16m+2,\ w_2(BC)=1,\ w_2(CD)=16m+6,\ w_2(EF)=5,$$

which is of the form (4.5) with $r=2$. In each of the cases described above, by what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

(b) If $w_1(B)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$ and $w_1(BC)=0$, P_2, after the first round, is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1(B)=16m+\ell$, $w_1(CD)=16m+6$ and $w_1(EF)=12$, and P_2 wins by Lemma 4.13.

(c) Finally, if $w_1(B)=16n+\ell_1$ for some $n<m$ and $\ell_1\in\{0,1,\ldots,15\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1=n$, $m_2=m$ and $\ell_2=6$. Since $n<m \implies m_1\neq m_2$, hence P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF)=k\leqslant 11$. If $k\in\{8,9,10,11\}$, then P_2 removes weight $(12-k)$ from B in the second round, leaving P_1 with

$$w_2(B)=16m+(k-8),\ w_2(BC)=2,\ w_2(CD)=16m+6,\ w_2(EF)=k,$$

which is of the form given by (4.4) with either $r=0$ (when $k=8$) or $r=1$ (when $k=9$) or $r=3$ (when $k=11$), or of the form given by (4.7) with $r=2$ and $s=2$ (when $k=10$). If $k=7$, then P_1 removes weight 4 from B and weight 1 from BC in the first round, leaving P_1 with

$$w_2 \quad B) = 16m,\ w_2\ BC) = 1,\ w_2\ CD) = 16m+6,\ w_2\ EF) = 7,$$

which is of the form (4.4) with $r=2$. By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we conclude that P_1 loses. If $k \in \{1,2,3,4,6\}$, the configuration after the first round is of the form stated in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = 6$, and $w_1\ EF) = k$ satisfying either $\min\{\ell_1, \ell_2\} \geqslant k$ or $k \in \{\ell_1, \ell_2\}$, so that P_2 wins by Lemma 4.9. Finally, if $k=5$, then P_2 removes weight 6 from the edge CD and weight 1 from the edge BC in the second round, leaving P_1 with

$$w_2 \quad B) = 16m+4,\ w_2\ BC) = 1,\ w_2\ CD) = 16m,\ w_2\ EF) = 5,$$

which is of the form (4.4) with $r=0$, and by what we have already proved in §7.4, we know that P_1 loses. If $k=0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of B and CD, so that $w_1\ BC) = t \in \{0,1\}$. If $t=1$, P_2 removes weight 9 from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 4\ 4m+1),\ w_2\ BC) = 1,\ w_2\ CD) = 4\ 4m+1)+2,\ w_2\ EF) = 3,$$

which is of the form (4.4) with $r=0$, and by what we have already proved in §7.4, we conclude that P_1 loses. If $t=0$, P_2 is left, after the first round, with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 16m+4$, $w_1\ CD) = 16m+6$ and $w_1\ EF) = 12$, and P_2 wins by Lemma 4.13.

This concludes the proof of our claim that the configuration in (8.10) is losing.

The base cases corresponding to (4.5) with $r=4$ and $s=2$ are obtained by setting $k=2$ and $k=3$, which yield, respectively, the configurations

$$w_0 \quad B) = 4m+4 = 4\ m+1),\ w_0\ BC) = 2,\ w_0\ CD) = 4m+4 = 4\ m+1),\ w_0\ EF) = 2,$$
$$w_0 \quad B) = 4m+4 = 4\ m+1),\ w_0\ BC) = 2,\ w_0\ CD) = 4m+5 = 4\ m+1)+1,\ w_0\ EF) = 3,$$

each of which is of the form (4.4) with $r=0$, and by what we have proved in §7.4, we know that P_1 loses.

Suppose, for some $K \in$, we have proved that any configuration that is either of the form (4.5) or of the form (4.6) with $r=4$ and $s=2$, and with $w_0\ EF) = k \leqslant K$, is losing.

8.10.1. *When $K \geqslant 9$ and $K \equiv i \bmod 8$ for some $i \in \{1,2\}$.* For $m \in \ _0$, we consider the configuration

$$w_0 \quad B) = 2^{R+1}m+4,\ w_0\ BC) = 2,\ w_0\ CD) = 2^{R+1}m+\ K+3),\ w_0\ EF) = \ K+1), \tag{8.11}$$

where $R = f\ K+1)$. We consider the first round of the game played on this initial configuration:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. This can be divided into the following subcases:

(a) Suppose $w_1\ CD) = 2^{R+1}m+\ K+2)$. If $K \equiv 1 \bmod 8$, so that $K = 8n+1$ for some $n \in$, P_2 removes weight 3 from B, and weight $w_1\ BC)$ from BC, in the second round, leaving P_1 with the galaxy graph consisting of B, CD and EF, where $w_2 \quad B) = 2^{R+1}m+1$, $w_2\ CD) = 2^{R+1}m+\ 8n+3)$ and $w_2\ EF) = \ 8n+2)$, and P_1 loses by Theorem 2.3. If $K \equiv 2 \bmod 8$ and $w_1\ BC) = 1$, P_2 removes weight 2 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m+K+2,\ w_2\ EF) = K \quad 1,$$

which is of the form (4.5) with $r=4$ and $s=1$, and by what we have proved in §8.9, we know that P_1 loses. If $w_1\ BC)=2$, then P_2 removes weight 1 from EF in the second round, so that

$$w_2\quad B)=2^{R+1}m+4,\ w_2\ BC)=2,\ w_2\ CD)=2^{R+1}m+\ K+2),\ w_2\ EF)=K,$$

which is of the form (4.5) with $r=4$ and $s=2$, so that P_1 loses by our induction hypothesis. Suppose $w_1\ CD)=2^{R+1}m+\ K+2)$ with $K\equiv 2 \bmod 8$, and $w_1\ BC)=0$. Writing $K=8n+2$ for some $n\in\quad$, P_2, after the first round, is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, where $w_1\quad B)=2^{R+1}m+4$, $w_1\ CD)=2^{R+1}m+\ 8n+4)$ and $w_1\ EF)=\ 8n+3)$, and she wins by Theorem 2.3.

(b) Suppose $w_1\ CD)=2^{R+1}m+\ K+1)$. Then, irrespective of what $w_1\ BC)$ is, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1=4$, $\ell_2=\ K+1)$ and $w_1\ EF)=\ K+1)$. Since $w_1\ EF)\in\{\ell_1,\ell_2\}$, P_2 wins by Lemma 4.9.

(c) If $w_1\ CD)=2^{R+1}m+K$ and $w_1\ BC)=t\in\{1,2\}$, P_2 removes weight 4 from $\quad B$ and weight $\{w_1\ BC)\quad 1\}$ from BC in the second round, leaving P_1 with

$$w_2\quad B)=2^{R+1}m,\ w_2\ BC)=1,\ w_2\ CD)=2^{R+1}m+K,\ w_2\ EF)=\ K+1),$$

which is of the form given by (4.4) with $r=0$, and by what we have already proved in §7.4, we know that P_1 loses. If $w_1\ CD)=2^{R+1}m+K$ and $w_1\ BC)=0$, P_2, after the first round, is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, with $w_1\quad B)=2^{R+1}m+4$, $w_1\ CD)=2^{R+1}m+K$ and $w_1\ EF)=\ K+1)$. Writing $K=8n+i$ for some $n\in\quad$ and $i\in\{1,2\}$, we see that the triple $\ 2^{R+1}m+3, 2^{R+1}m+K, K+1)$ is balanced when $i=1$, and the triple $\ 2^{R+1}m+1, 2^{R+1}m+K, K+1)$ is balanced when $i=2$, so that P_2 wins by Theorem 2.3.

(d) If $w_1\ CD)=2^{R+1}m+L$ for some $L\in\{4,5,\quad,K\quad 6\}$, with $L\equiv j \bmod 8$ for some $j\in\{0,1,2,3,6,7\}$, and $w_1\ BC)=2$, P_2 removes weight $\ K+1)\qquad L+6)$ from the edge EF in the second round, leaving P_1 with

$$w_2\quad B)=2^{f\ L+6)+1}2^{R\quad f\ L+6)}m+4,\ w_2\ BC)=2,$$
$$w_2\ CD)=2^{f\ L+6)+1}2^{R\quad f\ L+6)}m+L,\ w_2\ EF)=\ L+6),$$

which is of the form (4.6) with $r=4$ and $s=2$, and since $w_2\ EF)=\ L+6)\leqslant K$, P_1 loses by our induction hypothesis.

(e) If $w_1\ CD)=2^{R+1}m+L$ for some $L\in\{4,5,\quad,K+2\}$, with $L\equiv j \bmod 8$ for some $j\in\{4,5\}$, and $w_1\ BC)=2$, P_2 removes weight $\ K+1)\qquad L\quad 2)$ from the edge EF in the second round, leaving P_1 with

$$w_2\quad B)=2^{f\ L\quad 2)+1}2^{R\quad f\ L\quad 2)}m+4,\ w_2\ BC)=2,$$
$$w_2\ CD)=2^{f\ L\quad 2)+1}2^{R\quad f\ L\quad 2)}m+L,\ w_2\ EF)=\ L\quad 2),$$

which is of the form (4.5) with $r=4$ and $s=2$, and since $w_2\ EF)=\ L\quad 2)\leqslant K$, P_1 loses by our induction hypothesis. This case covers $L=\ K\quad 5)$ since $K\equiv i \bmod 8$ for $i\in\{1,2\}$.

(f) If $w_1\ CD)=2^{R+1}m+L$ for some $L\in\{K\quad 4, K\quad 3, K\quad 2, K\quad 1\}$, and $w_1\ BC)=2$, P_2 removes weight $\ L+5\quad K)$ from the edge $\quad B$ in the second round, leaving P_1 with

$$w_2\quad B)=2^{R+1}m+\ K\quad L\quad 1),\ w_2\ BC)=2,\ w_2\ CD)=2^{R+1}m+L,\ w_2\ EF)=\ K+1),$$

which is of the form given by (4.4) with either $r=0$ (when $L=K\quad 1$) or $r=1$ (when $L=K\quad 2$) or $r=3$ (when $L=K\quad 4$), or of the form (4.7) with $r=2$ and $s=2$ (when $L=K\quad 3$). By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we know that P_1 loses.

(g) If $w_1(CD) = 2^{R+1}m+L$ for some $L \in \{4,5,\ldots,K-5\}$ with $L \equiv j \bmod 8$ for some $j \in \{0,1,2,3,7\}$, and $w_1(BC) = 1$, P_2 removes weight $(K+1) - f(L+5)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+5)+1}2^{R-f(L+5)}m+4,\ w_2(BC) = 1,$$

$$w_2(CD) = 2^{f(L+5)+1}2^{R-f(L+5)}m+L,\ w_2(EF) = f(L+5),$$

which is of the form (4.6) with $r=4$ and $s=1$, and by what we have already proved in §8.9, we know that P_1 loses.

(h) If $w_1(CD) = 2^{R+1}m+L$ for some $L \in \{4,5,\ldots,K+2\}$, with $L \equiv j \bmod 8$ for some $j \in \{4,5,6\}$, and $w_1(BC) = 1$, P_2 removes weight $(K+1) - f(L-3)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-3)+1}2^{R-f(L-3)}m+4,\ w_2(BC) = 1,$$

$$w_2(CD) = 2^{f(L-3)+1}2^{R-f(L-3)}m+L,\ w_2(EF) = f(L-3),$$

which is of the form (4.5) with $r=4$ and $s=1$, and by what we have already proved in §8.9, we know that P_1 loses. This case covers $L = f(K-4)$, since $K \equiv i \bmod 8$ for some $i \in \{1,2\}$.

(i) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{K-3, K-2, K-1\}$, and $w_1(BC) = 1$, P_2 removes weight $(L+4-K)$ from the edge AB in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+(K-L),\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m+L,\ w_2(EF) = (K+1),$$

which is of the form (4.4) with either $r=3$ (when $L = K-3$) or $r=1$ (when $L = K-1$), or of the form (4.6) with $r=2$ and $s=1$ (when $L = K-2$). By what we have already proved in §8.5, §8.8 and §7.6, we conclude that P_1 loses.

(j) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{0,1,2,3\}$, and $w_1(BC) = t \in \{1,2\}$, P_2 removes weight $(K+1) - (4+t+L)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+4,\ w_2(BC) = t,\ w_2(CD) = 2^{R+1}m+L,\ w_2(EF) = (4+t+L),$$

which is of the form (4.4) with either $r=3$ (when $L=3$) or $r=1$ (when $L=1$) or $r=0$ (when $L=0$), or of the form (4.6) with $r=2$ and $s=1$ (when $L=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s=2$ (when $L=2$ and $t=2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

(k) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{0,1,\ldots,K-1\}$, and $w_1(BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+L$ and $w_1(EF) = (K+1)$. When $L \leqslant f(K-4)$, P_2 wins by Lemma 4.13. When $L = f(K-3)$, writing $K = 8(n+1)+i$ for some $n \in \mathbb{N}_0$ and $i \in \{1,2\}$, we see that the triple $\left(2^{R+1}m+4, 2^{R+1}m+8n+5+i, 8n+1+i\right)$ is balanced, and hence, P_2 wins by Theorem 2.3. When $L = f(K-2)$ and $K \equiv 1 \bmod 8$, writing $K = 8(n+1)+1$ for some $n \in \mathbb{N}_0$, we see that the triple $\left(2^{R+1}m+4, 2^{R+1}m+8n+7, 8n+3\right)$ is balanced, whereas if $K \equiv 2 \bmod 8$, writing $K = 8(n+1)+2$ for some $n \in \mathbb{N}_0$, we see that the triple $\left(2^{R+1}m+3, 2^{R+1}m+8n+8, 8n+11\right)$ is balanced – consequently, P_2 wins by Theorem 2.3. When $L = f(K-1)$, writing $K = 8(n+1)+i$ for some $n \in \mathbb{N}_0$ and $i \in \{1,2\}$, we see that the triple $\left(2^{R+1}m+2, 2^{R+1}m+8n+7+i, 8n+9+i\right)$ is balanced, so that P_2 wins by Theorem 2.3.

(l) Finally, if $w_1(CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from AB and a non-negative integer weight from BC in the first round. We have the following possibilities:

(a) If $w_1 \quad B) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,2,3\}$ and $w_1 \; BC) = t \in \{1,2\}$, P_2 removes weight $\; 2+\ell+t)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+\ell,\; w_2 \; BC) = t,$$
$$w_2 \; CD) = 2^{R+1}m+ \; K+1) \quad \ell \quad t,\; w_2 \; EF) = \; K+1),$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.6) with $r=2$ and $s=1$ (when $\ell=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s=2$ (when $\ell=2$ and $t=2$).

(b) If $w_1 \quad B) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,2,3\}$ and $w_1 \; BC) = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges $\; B$, CD and EF, with $w_1 \quad B) = 2^{R+1}m+\ell$, $w_1 \; CD) = 2^{R+1}m+ \; K+3)$ and $w_1 \; EF) = \; K+1)$. Writing $K = 8n+i$ for $i \in \{1,2\}$ and some $n \in \quad$, the resulting triple is $\; 2^{R+1}m+\ell, 2^{R+1}m+8n+i+3, 8n+i+1)$, and this is winning by Part (B2) of Theorem 4.8.

(c) Finally, if $w_1 \quad B) = 2^{R+1}n+\ell_1$ for some $n<m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = K+3$. Since $n<m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1 \; EF) = k \leqslant K$. If $k \in \{12,13, \quad ,K\}$ and $k \equiv j \bmod 8$ for some $j \in \{0,1,4,5,6,7\}$, then P_2 removes weight $\; K+9 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \; k)+1}2^{R \quad f \; k)}m+4,\; w_2 \; BC) = 2,$$
$$w_2 \; CD) = 2^{f \; k)+1}2^{R \quad f \; k)}m+ \; k \quad 6),\; w_2 \; EF) = k,$$

which is of the form (4.6) with $r=4$, $s=2$ and $k \leqslant K$, so that P_1 loses by our induction hypothesis. If $k \in \{10,11, \quad ,K\}$ and $k \equiv j \bmod 8$ for some $j \in \{2,3\}$, then P_2 removes weight $\; K+1 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f \; k)+1}2^{R \quad f \; k)}m+4,\; w_2 \; BC) = 2,$$
$$w_2 \; CD) = 2^{f \; k)+1}2^{R \quad f \; k)}m+ \; k+2),\; w_2 \; EF) = k,$$

which is of the form (4.5) with $r=4$, $s=2$ and $k \leqslant K$, so that P_1 loses by our induction hypothesis.

If $k \in \{1,2,3,4\}$, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = \; K+3)$ and $w_1 \; EF) = k \leqslant \min\{\ell_1,\ell_2\}$. Thus, P_2 wins by Lemma 4.9. If $k=5$, then the configuration after the first round is of the form mentioned in Lemma 4.11, with $\ell_1 = 4$, $\ell_2 = \; K+3)$, $w_1 \; EF) = k = 5 > \min\{\ell_1,\ell_2\}$ and $w_1 \; BC) = 2 > k \quad \min\{\ell_1,\ell_2\}$, so that P_2 wins by Lemma 4.11. If $k \in \{6,7,8,9\}$, then P_2 removes weight $\; K+9 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\; w_2 \; BC) = 2,\; w_2 \; CD) = 2^{R+1}m+ \; k \quad 6),\; w_2 \; EF) = k,$$

which is of the form (4.4) with either $r=0$ (when $k=6$) or $r=1$ (when $k=7$) or $r=3$ (when $k=9$), or of the form (4.7) with $r=2$ and $s=2$ (when $k=8$). By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we know that P_1 loses. If $k=0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from the edge BC in the first round, without disturbing the edge-weights of $\; B$ and CD, so that $w_1 \; BC) = t \in \{0,1\}$. If $t=1$, P_2 removes weight 1 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\; w_2 \; BC) = 1,\; w_2 \; CD) = 2^{R+1}m+ \; K+3),\; w_2 \; EF) = K,$$

which is of the form (4.5) with $r = 4$ and $s = 1$, and by what we have already proved in §8.9, we conclude that P_1 loses. If $t = 0$, P_2 is left with a galaxy graph after the first round, consisting of the edges AB, CD and EF, with $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+(K+3)$ and $w_1(EF) = (K+1)$. Writing $K = 8n+i$ for some $n \in \mathbb{N}$ and $i \in \{1,2\}$, we see that the triple $\left(2^{R+1}m+4, 2^{R+1}m+8n+3+i, 8n-1+i\right)$ is balanced, so that P_2 wins by Theorem 2.3.

This concludes the proof of our claim that the configuration in (8.11) is losing on H_1.

8.10.2. *When* $K \geqslant 12$ *and* $K \equiv i \bmod 8$ *for some* $i \in \{0,3,4,5,6,7\}$. For $m \in \mathbb{N}_0$, we consider

$$w_0(AB) = 2^{R+1}m+4,\ w_0(BC) = 2,\ w_0(CD) = 2^{R+1}m+(K-5),\ w_0(EF) = (K+1), \tag{8.12}$$

where $R = f(K+1)$. The first round of the game played on this initial configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. The following possibilities may arise:

(a) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{4,5,\ldots,K-6\}$, with $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,6,7\}$, and $w_1(BC) = 2$, P_2 removes weight $(K-L-5)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+6)+1}2^{R-f(L+6)}m+4,\ w_2(BC) = 2,$$
$$w_2(CD) = 2^{f(L+6)+1}2^{R-f(L+6)}m+L,\ w_2(EF) = (L+6),$$

which is of the form (4.6) with $r = 4$, $s = 2$ and $w_2(EF) = (L+6) \leqslant K$, so that P_1 loses by our induction hypothesis.

(b) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{4,5,\ldots,K-6\}$ with $L \equiv j \bmod 8$ for some $j \in \{4,5\}$, and $w_1(BC) = 2$, P_2 removes weight $(K+3-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-2)+1}2^{R-f(L-2)}m+4,\ w_2(BC) = 2,$$
$$w_2(CD) = 2^{f(L-2)+1}2^{R-f(L-2)}m+L,\ w_2(EF) = (L-2),$$

which is of the form (4.5) with $r = 4$, $s = 2$ and $w_2(EF) = (L-2) \leqslant (K-8) < K$, so that P_1 loses by our induction hypothesis.

(c) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{4,5,\ldots,K-6\}$, with $L \equiv j \bmod 8$ for some $j \in \{0,1,2,3,7\}$, and $w_1(BC) = 1$, P_2 removes weight $(K-4-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+5)+1}2^{R-f(L+5)}m+4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(L+5)+1}2^{R-f(L+5)}m+L,\ w_2(EF) = (L+5),$$

which is of the form (4.6) with $r = 4$ and $s = 1$, and P_1 loses by what we have proved in §8.9.

(d) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{4,5,\ldots,K-6\}$, with $L \equiv j \bmod 8$ for some $j \in \{4,5,6\}$, and $w_1(BC) = 1$, P_2 removes weight $(K+4-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-3)+1}2^{R-f(L-3)}m+4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(L-3)+1}2^{R-f(L-3)}m+L,\ w_2(EF) = (L-3),$$

which is of the form (4.5) with $r = 4$ and $s = 1$, and P_1 loses by what we have proved in §8.9.

(e) If $w_1(CD) = 2^{R+1}m+L$ for $L \in \{0,1,2,3\}$ and $w_1(BC) = t \in \{1,2\}$, P_2 removes weight $(K-3-t-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+4,\ w_2(BC) = t,\ w_2(CD) = 2^{R+1}m+L,\ w_2(EF) = 4+t+L,$$

which is of the form (4.4) with either $r = 0$ (when $L = 0$) or $r = 1$ (when $L = 1$) or $r = 3$ (when $L = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $L = 2$ and $t = 1$), or of the form (4.7) with $r = 2$ and $s = 2$ (when $L = 2$ and $t = 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

(f) If $w_1\ CD) = 2^{R+1}m + L$ for some $L \in \{0, 1, \quad, K \quad 6\}$, and $w_1\ BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m + 4$, $w_1\ CD) = 2^{R+1}m + L$ and $w_1\ EF) = \ K + 1)$, and she wins by Lemma 4.13.

(g) Finally, if $w_1\ CD) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell_1 \in \{0, 1, \quad, 2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\quad B$, and a non-negative integer weight from BC, in the first round. The following possibilities may arise:

(a) If either $w_1 \quad B) = 2^{R+1}m + \ell$ for some $\ell \in \{0, 1, 3\}$ and $w_1\ BC) = t \in \{1, 2\}$, or $w_1 \quad B) = 2^{R+1}m + 2$ and $w_1\ BC) = t = 2$, or $w_1 \quad B) = 2^{R+1}m + 2$ and $w_1\ BC) = t = 1$ and $K \equiv i \bmod 8$ for $i \in \{0, 4, 5, 6\}$, P_2 removes weight $\ 6 \quad \ell \quad t)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + \ell,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m + \ K \quad 5),\ w_2\ EF) = K + \ell + t \quad 5,$$

which is of the form (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$) or $r = 3$ (when $\ell = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $\ell = 2$ and $t = 1$, or of the form (4.7) with $r = 2$ and $s = 2$ (when $\ell = 2$ and $t = 2$). By what we have proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_1 loses.

(b) If $w_1 \quad B) = 2^{R+1}m + 2$ and $w_1\ BC) = t = 1$, where $K \equiv i \bmod 8$ for $i \in \{3, 7\}$, P_2 removes weight 7 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 2,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m + \ K \quad 5),\ w_2\ EF) = \ K \quad 6),$$

which is of the form (4.5) with $r = 2$ and $s = 1$, and by what we have already proved in §7.6, we know that P_1 loses.

(c) If $w_1 \quad B) = 2^{R+1}m + \ell$ for some $\ell \in \{0, 1, 2, 3\}$, and $w_1\ BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m + \ell$, $w_1\ CD) = 2^{R+1}m + \ K \quad 5)$ and $w_1\ EF) = \ K + 1)$, and she wins by Lemma 4.13.

(d) Finally, if $w_1 \quad B) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell_1 \in \{0, 1, \quad, 2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = K \quad 5$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$. If $k \in \{12, 13, \quad, K\}$ and $k \equiv j \bmod 8$ for some $j \in \{0, 1, 4, 5, 6, 7\}$, P_2 removes weight $\ K + 1 \quad k)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ k)+1}2^{R \quad f\ k)}m + 4,\ w_2\ BC) = 2,$$
$$w_2\ CD) = 2^{f\ k)+1}2^{R \quad f\ k)}m + \ k \quad 6),\ w_2\ EF) = k,$$

which is of the form (4.6) with $r = 4$, $s = 2$ and $w_2\ EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. Note that this case covers (a) $k = \ K \quad 7)$, (b) $k = \ K \quad 6)$ and $K \equiv i \bmod 8$ for $i \in \{3, 4, 5, 6, 7\}$. If $k = \ K \quad 6)$ and $K \equiv 0 \bmod 8$, writing $K = 8\ n + 1)$ for some $n \in \quad_0$, P_2 removes weight 3 from $\quad B$, and the entire BC, in the second round, leaving P_1 with a galaxy graph consisting of the edges $\quad B$, CD and EF, with $w_2 \quad B) = 2^{R+1}m + 1$, $w_2\ CD) = 2^{R+1}m + 8n + 3$ and $w_2\ EF) = 8n + 2$, and P_1 loses by Theorem 2.3.

If $k \in \{10, 11, \quad, K \quad 8\}$ and $k \equiv j \bmod 8$ for some $j \in \{2, 3\}$, P_2 removes weight $\ K \quad k \quad 7)$ from the edge CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ k)+1}2^{R \quad f\ k)}m + 4,\ w_2\ BC) = 2,$$

$$w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m + (k+2),\ w_2(EF) = k,$$

which is of the form (4.5) with $r = 4$, $s = 2$ and $w_2(EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis.

If $k \in \{1,2,3,4,K-5\}$, then the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = (K-5)$ and $w_1(EF) = k$, so that either $\min\{\ell_1,\ell_2\} \geqslant k$ or $k \in \{\ell_1,\ell_2\}$. Consequently, P_2 wins by Lemma 4.9. If $k \in \{K-3,\ldots,K\}$, then P_2 removes $(K+1-k)$ from $\;B$ in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + k - (K-3),\ w_2(BC) = 2,\ w_2(CD) = 2^{R+1}m + (K-5),\ w_2(EF) = k,$$

which is of the form (4.4) with either $r = 0$ (when $k = K-3$) or $r = 1$ (when $k = K-2$) or $r = 3$ (when $k = K$), or of the form (4.7) with $r = 2$ and $s = 2$ (when $k = K-1$). If $k = (K-4)$, P_2 removes weight 4 from $\;B$ and weight 1 from BC in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + (K-5),\ w_2(EF) = (K-4),$$

which is of the form (4.4) with $r = 0$. By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we know that P_1 loses. If $k \in \{5,6,8\}$, P_2 removes weight $(K-k)$ from CD and weight 1 from BC in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + 4,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + (k-5),\ w_2(EF) = k,$$

which is of the form (4.4) with either $r = 0$ (when $k = 5$) or $r = 1$ (when $k = 6$) or $r = 3$ (when $k = 8$); if $k = 7$, P_2 removes weight $(K-6)$ from CD in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + 4,\ w_2(BC) = 2,\ w_2(CD) = 2^{R+1}m + 1,\ w_2(EF) = 7,$$

which is of the form (4.4) with $r = 1$; if $k = 9$, P_2 removes weight $(K-8)$ from CD in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + 4,\ w_2(BC) = 2,\ w_2(CD) = 2^{R+1}m + 3,\ w_2(EF) = 9,$$

which is of the form (4.4) with $r = 3$. By what we have already proved in §7.4, §8.5 and §8.8, we conclude that P_1 loses. If $k = 0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from the edge BC in the first round, without disturbing the edge-weights of $\;B$ and CD. If $w_1(BC) = 1$, and $K \equiv i \bmod 8$ for some $i \in \{0,4,5,6,7\}$, then P_2 removes weight 1 from EF in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + 4,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + (K-5),\ w_2(EF) = K,$$

which is of the form (4.6) with $r = 4$ and $s = 1$, and by what we have already proved in §8.9, we conclude that P_1 loses. If $w_1(BC) = 1$ and $K \equiv 1 \bmod 8$, P_2 removes weight 9 from EF in the second round, leaving P_1 with

$$w_2(\;B) = 2^{R+1}m + 4,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m + (K-5),\ w_2(EF) = (K-8),$$

which is of the form (4.5) with $r = 4$ and $s = 1$, and by what we have already proved in §8.9, we conclude that P_1 loses. If $w_1(BC) = 0$, then P_2, after the first round, is left with a galaxy graph consisting of the edges $\;B$, CD and EF, where $w_1(\;B) = 2^{R+1}m + 4$, $w_1(CD) = 2^{R+1}m + (K-5)$ and $w_1(EF) = (K+1)$, and she wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.12) is losing, and thereby, the proof of our claim that aconfiguration that is either of the form (4.5) or of the form (4.6), with $r = 4$ and $s = 2$, is losing.

8.11. **Proof that configurations satisfying** (4.5) **or** (4.6)**, with** $r=4$ **and** $s=3$**, are losing.** The base case corresponding to (4.6) with $r=4$ and $s=3$ is obtained by setting $k=12$, which yields the configuration

$$w_0 \quad B) = 16m+4,\ w_0\ BC) = 3,\ w_0\ CD) = 16m+5,\ w_0\ EF) = 12 \tag{8.13}$$

The first round of the game played on this configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. This can be further divided into the following possibilities:

(a) If $w_1\ CD) = 16m+\ell$ for $\ell \in \{0,1,2,3\}$ and $w_1\ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ 8\ \ \ell\ \ t)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 16m+4,\ w_2\ BC) = t,\ w_2\ CD) = 16m+\ell,\ w_2\ EF) = 4+\ell+t,$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.6) with $r=2$ and $s=1$ (when $\ell=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s \geqslant 2$ (when $\ell=2$ and $t \geqslant 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_1 loses.

(b) If $w_1\ CD) = 16m+4$ and $w_1\ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ 12\ \ t)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 4\ \ 4m+1),\ w_2\ BC) = t,\ w_2\ CD) = 4\ \ 4m+1),\ w_2\ EF) = t,$$

which is of the form (4.4) with $r=0$, so that P_1 loses by what we have already proved in §7.4.

(c) If $w_1\ CD) = 16m+\ell$ for some $\ell \in \{0,1,2,3,4\}$ and $w_1\ BC) = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1\ \ B) = 16m+4$, $w_1\ CD) = 16m+\ell$ and $w_1\ EF) = 12$, and she wins by Lemma 4.13.

(d) Finally, if $w_1\ CD) = 16n+\ell_1$ for some $n<m$ and $\ell_1 \in \{0,1,\ \ ,15\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n<m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\ B$, and a non-negative integer weight from BC, in the first round. We again consider the various possibilities separately:

(a) If $w_1\ \ B) = 16m+\ell$ for $\ell \in \{0,1,2,3\}$ and $w_1\ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ 7\ \ \ell\ \ t)$ from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 16m+\ell,\ w_2\ BC) = t,\ w_2\ CD) = 16m+5,\ w_2\ EF) = 5+\ell+t,$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.6) with $r=2$ and $s=1$ (when $\ell=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s \geqslant 2$ (when $\ell=2$ and $t \geqslant 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

(b) If $w_1\ \ B) = 16m+\ell$ for $\ell \in \{0,1,2,3\}$ and $w_1\ BC) = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1\ \ B) = 16m+\ell$, $w_1\ CD) = 16m+5$ and $w_2\ EF) = 12$, and she wins by Lemma 4.13.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant 11$. If $k \in \{8,9,10,11\}$, P_2 removes weight $\ 12\ \ k)$ from $\ B$ in the second round, leaving P_1 with

$$w_2 \quad B) = 16m+\ \ k\ \ 8),\ w_2\ BC) = 3,\ w_2\ CD) = 16m+5,\ w_2\ EF) = k,$$

which is of the form (4.4) with either $r=0$ (when $k=8$) or $r=1$ (when $k=9$) or $r=3$ (when $k=11$), or of the form (4.7) with $r=2$ and $s=3$ (when $k=10$). If $k \in \{6,7\}$, P_2 removes weight 4 from $\ B$ and weight $\ 8\ \ k)$ from BC in the second round, leaving P_1 with

$$w_2 \quad B) = 16m,\ w_2\ BC) = \ \ k\ \ 5),\ w_2\ CD) = 16m+5,\ w_2\ EF) = k,$$

which is of the form (4.4) with $r = 0$. By what we have proved in §7.4, §8.5, §8.8 and §7.7, we conclude that P_1 loses. If $k \in \{1,2,3,4,5\}$, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = 5$, and $w_1(EF) = k$ satisfying either $\min\{\ell_1, \ell_2\} \geqslant k$ or $k \in \{\ell_1, \ell_2\}$, so that P_2 wins by Lemma 4.9. If $k = 0$, P_2 wins by Remark 4.12.

(iv) Finally, suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of AB and CD, so that $w_1(BC) = t \in \{0,1,2\}$. For $t \in \{1,2\}$, P_2 removes weight $(11 - t)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 4(4m+1),\ w_2(BC) = t,\ w_2(CD) = 4(4m+1)+1,\ w_2(EF) = t+1,$$

which is of the form (4.4) with $r = 0$. By what we have proved in §7.4, we conclude that P_1 loses. If $t = 0$, P_2 is left with a galaxy graph at the end of the first round, consisting of the edges AB, CD and EF, where $w_1(AB) = 16m+4$, $w_1(CD) = 16m+5$ and $w_1(EF) = 12$, and she wins by Lemma 4.13.

This proves our claim that the configuration in (8.13) is losing on H_1. The base case corresponding to (4.5) with $r = 4$ and $s = 3$ is obtained by setting $k = 3$, which yields the configuration

$$w_0(AB) = 16m+4 = 4(4m+1),\ w_0(BC) = 3,\ w_0(CD) = 16m+4 = 4(4m+1),\ w_0(EF) = 3,$$

which is losing since it is of the form (4.4) with $r = 0$.

Suppose, for some $K \in \mathbb{N}$, we have shown that a configuration that is either of the form (4.5) or of the form (4.6), with $r = 4$ and $s = 3$, is losing as long as $w_0(EF) = k \leqslant K$.

8.11.1. *When $K \geqslant 10$ and $K \equiv 2 \bmod 8$.* For $m \in \mathbb{N}_0$, we consider the configuration

$$w_0(AB) = 2^{R+1}m+4,\ w_0(BC) = 3,\ w_0(CD) = 2^{R+1}m+K+2,\ w_0(EF) = K+1, \tag{8.14}$$

where $R = f(K+1)$. The first round of the game played on this initial configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD and a non-negative integer weight from BC in the first round. The following possibilities may arise:

(a) If $w_1(CD) = 2^{R+1}m + (K+1)$, the configuration, after the first round, is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = (K+1)$ and $w_1(EF) = k = (K+1) \in \{\ell_1, \ell_2\}$, so that P_2 wins by Lemma 4.9.

(b) If $w_1(CD) = 2^{R+1}m+K$ and $w_1(BC) = t \in \{1,2,3\}$, P_2 removes weight 4 from AB, and weight $(t-1)$ from BC, in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m+K,\ w_2(EF) = K+1,$$

which is of the form (4.4) with $r = 0$, and P_1 loses by what we have already proved in §7.4. If $w_1(CD) = 2^{R+1}m+K$ and $w_1(BC) = 0$, writing $K = 8n+2$ for some $n \in \mathbb{N}$, we see that P_2 is left, at the end of the first round, with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+8n+2$ and $w_1(EF) = 8n+3$, and since the triple $(2^{R+1}m+1, 2^{R+1}m+8n+2, 8n+3)$ is balanced, P_2 wins by Theorem 2.3.

(c) Suppose $w_1(CD) = 2^{R+1}m + (K-1)$. If $w_1(BC) = t \in \{2,3\}$, P_2 removes weight $(t-2)$ from the edge BC and weight 4 from the edge AB in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m,\ w_2(BC) = 2,\ w_2(CD) = 2^{R+1}m+(K-1),\ w_2(EF) = (K+1),$$

which is of the form (4.4) with $r = 0$. If $w_1(BC) = 1$, P_2 removes weight 3 from AB in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+1,\ w_2(BC) = 1,\ w_2(CD) = 2^{R+1}m+(K-1),\ w_2(EF) = (K+1),$$

which is of the form (4.4) with $r = 1$. By what we have already proved in §7.4 and §8.5, we know that P_1 loses. If $w_1(BC) = 0$, writing $K = 8n+2$ for some $n \in \mathbb{N}$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1 \ CD) = 2^{R+1}m+8n+1$ and $w_1 \ EF) = 8n+3$, and since the triple $\ 2^{R+1}m+2, 2^{R+1}m+8n+1, 8n+3)$ is balanced, P_2 wins by Theorem 2.3.

(d) Suppose $w_1 \ CD) = 2^{R+1}m + \ K \quad 2)$. If $w_1 \ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ t+1)$ from the edge B in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + \ 3 \quad t),\ w_2 \ BC) = t,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 2),\ w_2 \ EF) = \ K+1),$$

which is of the form (4.4) with either $r = 0$ (when $t = 3$) or $r = 1$ (when $t = 2$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $t = 1$). By what we have already proved in §7.4, §8.5 and §7.6, we conclude that P_1 loses. If $w_1 \ BC) = 0$, writing $K = 8n+2$ for some $n \in \quad$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1 \ CD) = 2^{R+1}m+8n$ and $w_1 \ EF) = 8n+3$, and since the triple $\ 2^{R+1}m+3, 2^{R+1}m+8n, 8n+3)$ is balanced, P_2 wins by Theorem 2.3.

(e) Suppose $w_1 \ CD) = 2^{R+1}m + \ K \quad 3)$. If $w_1 \ BC) = t \in \{1,2,3\}$, then P_2 removes weight t from the edge B in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + \ 4 \quad t),\ w_2 \ BC) = t,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 3),\ w_2 \ EF) = \ K+1),$$

which is of the form (4.4) with either $r = 1$ (when $t = 3$) or $r = 3$ (when $t = 1$), or of the form (4.7) with $r = 2$ and $s = 2$ (when $t = 2$). By what we have already proved in §8.5, §8.8 and §7.7, we conclude that P_1 loses. If $w_1 \ BC) = 0$, writing $K = 8 \ n+1)+2$ for some $n \in \ _0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1 \ CD) = 2^{R+1}m+8n+7$ and $w_1 \ EF) = 8n+11$, and since the triple $\ 2^{R+1}m+4, 2^{R+1}m+8n+7, 8n+3)$ is balanced, P_2 wins by Theorem 2.3.

(f) If $w_1 \ CD) = 2^{R+1}m+L$ for some $L \in \{0,1, \quad ,K \quad 4\}$, and $w_1 \ BC) = 0$, then, at the end of the first round, P_2 is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1 \ CD) = 2^{R+1}m+L$ and $w_1 \ EF) = \ K+1)$, and she wins by Lemma 4.13.

(g) Suppose $w_1 \ CD) = 2^{R+1}m + \ K \quad 4)$. If $w_1 \ BC) = t \in \{2,3\}$, P_2 removes weight $\ t \quad 1)$ from the edge B in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + \ 5 \quad t),\ w_2 \ BC) = t,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 4),\ w_2 \ EF) = \ K+1),$$

which is of the form (4.7) with $r = 2$ and $s = 3$ (when $t = 3$) or of the form (4.4) with $r = 3$ (when $t = 2$). If $w_1 \ BC) = 1$, P_2 removes 8 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2 \ BC) = 1,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 4),\ w_2 \ EF) = \ K \quad 7),$$

which is of the form (4.5) with $r = 4$ and $s = 1$. By what we have already proved in §7.7, §8.8 and §8.9, we conclude that P_1 loses in each of the above three cases.

(h) Suppose $w_1 \ CD) = 2^{R+1}m + \ K \quad 5)$. If $w_1 \ BC) = 3$, P_2 removes weight 1 from the edge B in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+3,\ w_2 \ BC) = 3,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 5),\ w_2 \ EF) = \ K+1),$$

which is of the form (4.4) with $r = 3$. If $w_1 \ BC) = 2$, P_2 removes weight 8 from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2 \ BC) = 2,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 5),\ w_2 \ EF) = \ K \quad 7),$$

which is of the form (4.5) with $r = 4$ and $s = 2$. If $w_1 \ BC) = 1$, P_2 removes weight 9 from the edge EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2 \ BC) = 1,\ w_2 \ CD) = 2^{R+1}m + \ K \quad 5),\ w_2 \ EF) = \ K \quad 8),$$

which is of the form (4.5) with $r=4$ and $s=1$. By what we have already proved in §8.8, §8.9 and §8.10, we conclude that P_1 loses in each of these cases.

(i) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-6\}$, with $L\equiv 4 \bmod 8$, and $w_1(BC)=3$, P_2 removes weight $(K+2-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L-1)+1}2^{R-f(L-1)}m+4,\ w_2(BC)=3,$$
$$w_2(CD)=2^{f(L-1)+1}2^{R-f(L-1)}m+L,\ w_2(EF)=L-1,$$

which is of the form (4.5) with $r=4$ and $s=3$, and since $w_2(EF)=(L-1)<K$, P_1 loses by our induction hypothesis. This case covers $w_1(CD)=2^{R+1}m+(K-6)$ and $w_1(BC)=3$.

(j) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-7\}$, with $L\equiv i \bmod 8$ for $i\in\{0,1,2,3,5,6,7\}$, and $w_1(BC)=3$, P_2 removes weight $(K-6-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L+7)+1}2^{R-f(L+7)}m+4,\ w_2(BC)=3,$$
$$w_2(CD)=2^{f(L+7)+1}2^{R-f(L+7)}m+L,\ w_2(EF)=L+7,$$

which is of the form (4.6) with $r=4$ and $s=3$, and since $w_2(EF)=(L+7)\leqslant K$, P_1 loses by our induction hypothesis.

(k) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-6\}$, with $L\equiv j \bmod 8$ for $j\in\{4,5\}$, and $w_1(BC)=2$, P_2 removes weight $(K+3-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L-2)+1}2^{R-f(L-2)}m+4,\ w_2(BC)=2,$$
$$w_2(CD)=2^{f(L-2)+1}2^{R-f(L-2)}m+L,\ w_2(EF)=(L-2),$$

which is of the form (4.5) with $r=4$ and $s=2$, and by what we have already proved in §8.10, we conclude that P_1 loses.

(l) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-6\}$, with $L\equiv j \bmod 8$ for $j\in\{0,1,2,3,6,7\}$, and $w_1(BC)=2$, P_2 removes weight $(K-5-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L+6)+1}2^{R-f(L+6)}m+4,\ w_2(BC)=2,$$
$$w_2(CD)=2^{f(L+6)+1}2^{R-f(L+6)}m+L,\ w_2(EF)=(L+6),$$

which is of the form (4.6) with $r=4$ and $s=2$, and by what we have already proved in §8.10, we know that P_1 loses.

(m) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-6\}$, with $L\equiv j \bmod 8$ for $j\in\{4,5,6\}$, and $w_1(BC)=1$, P_2 removes weight $(K+4-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L-3)+1}2^{R-f(L-3)}m+4,\ w_2(BC)=1,$$
$$w_2(CD)=2^{f(L-3)+1}2^{R-f(L-3)}m+L,\ w_2(EF)=(L-3),$$

which is of the form (4.5) with $r=4$ and $s=1$, and by what we have already proved in §8.9, we know that P_1 loses.

(n) If $w_1(CD)=2^{R+1}m+L$ for $L\in\{4,5,\ldots,K-6\}$, with $L\equiv j \bmod 8$ for $j\in\{0,1,2,3,7\}$, and $w_1(BC)=1$, P_2 removes weight $(K-4-L)$ from EF in the second round, leaving P_1 with

$$w_2(B)=2^{f(L+5)+1}2^{R-f(L+5)}m+4,\ w_2(BC)=1,$$
$$w_2(CD)=2^{f(L+5)+1}2^{R-f(L+5)}m+L,\ w_2(EF)=(L+5),$$

which is of the form (4.6) with $r=4$ and $s=1$, and by what we have already proved in §8.9, we know that P_1 loses.

(o) If $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{0,1,2,3\}$ and $w_1\ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ K+1) \quad 4+t+L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m+L,\ w_2\ EF) = 4+t+L,$$

which is of the form (4.4) with either $r = 0$ (when $L = 0$) or $r = 1$ (when $L = 1$) or $r = 3$ (when $L = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $L = 2$ and $t = 1$), or of the form (4.7) with $r = 2$ and $s = t \geqslant 2$ (when $L = 2$ and $t \geqslant 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_1 loses in each of these cases.

(p) Finally, if $w_1\ CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\quad B$, and a non-negative integer weight from BC, in the first round. Once again, we consider the various possibilities:

(a) If $w_1 \quad B) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,2,3\}$, and $w_1\ BC) = t \in \{1,2,3\}$, P_2 removes weight $\ \ell+t+1)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+\ell,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m+K+1 \quad \ell \quad t,\ w_2\ EF) = K+1,$$

which is of the form (4.4) with $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$) or $r = 3$ (when $\ell = 3$), of the form (4.6) with $r = 2$ and $s = 1$ (when $\ell = 2$ and $t = 1$), or of the form (4.7) with $r = 2$ and $s \geqslant 2$ (when $\ell = 2$ and $t \geqslant 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_1 loses in each of these cases.

(b) If $w_1 \quad B) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,2,3\}$, and $w_1\ BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of $\quad B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m+\ell$, $w_1\ CD) = 2^{R+1}m+\ K+2)$ and $w_1\ EF) = \ K+1)$. Writing $K = 8n+2$ for some $n \in \quad$, since the triple $\ 2^{R+1}m+\ell, 2^{R+1}m+8n+3 \quad \ell, 8n+3)$ is balanced, P_2 wins by Theorem 2.3.

(c) Finally, if $w_1 \quad B) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = K+2$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1\ EF) = k \leqslant K$. If $k = 0$, P_2 wins by Remark 4.12. If $k \in \{1,2,3,4\}$, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$ and $\ell_2 = \ K+2)$, so that $w_1\ EF) = k$ satisfies the inequality $k \leqslant \min\{\ell_1,\ell_2\}$. Consequently, P_2 wins by Lemma 4.9. If $k \in \{5,6, \quad ,K\}$ and $k \equiv 3 \bmod 8$, then P_2 removes weight $\ K+1 \quad k)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = 3,\ w_2\ CD) = 2^{R+1}m+\ k+1),\ w_2\ EF) = k,$$

which is of the form (4.5) with $r = 4$, $s = 3$ and $w_2\ EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. If $k \in \{7,8, \quad ,K\}$ and $k \equiv j \bmod 8$ for some $j \in \{0,1,2,4,5,6,7\}$, then P_2 removes weight $\ K+9 \quad k)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = 3,\ w_2\ CD) = 2^{R+1}m+\ k \quad 7),\ w_2\ EF) = k, \tag{8.15}$$

which, when $k \geqslant 12$, is of the form (4.6) with $r = 4$, $s = 3$ and $w_2\ EF) = k \leqslant K$, so that P_1 loses by our induction hypothesis. If $k \in \{7,8,9,10\}$, the configuration in (8.15) is of the form (4.4) with either $r = 0$ (when $k = 7$) or $r = 1$ (when $k = 8$) or $r = 3$ (when $k = 10$), or of the form (4.7) with $r = 2$ and $s = 3$ (when $k = 9$). If $k \in \{5,6\}$, P_2 removes weight $\ K+2)$ from the edge CD and weight $\ 7 \quad k)$ from the edge BC in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = k \quad 4,\ w_2\ CD) = 2^{R+1}m,\ w_2\ EF) = k,$$

which is of the form (4.4) with $r = 0$. By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we know that P_1 loses in each of these cases.

(iv) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of $\ B$ and CD, so that $w_1\ BC) = t \in \{0,1,2\}$. If $t \in \{1,2\}$, P_2 removes weight $\ 3 \quad t)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m+4,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m+\ K+2),\ w_2\ EF) = \ K \quad 2+t),$$

which is of the form (4.5) with either $r = 4$ and $s = 2$ (when $t = 2$) or $r = 4$ and $s = 1$ (when $t = 1$). By what we have already proved in §8.9 and §8.10, we know that P_1 loses. If $t = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges $\ B$, CD and EF, where $w_1 \quad B) = 2^{R+1}m+4$, $w_1\ CD) = 2^{R+1}m+\ K+2)$ and $w_1\ EF) = \ K+1)$. Writing $K = 8n+2$ for some $n \in \quad$, since the triple $\ 2^{R+1}m+4, 2^{R+1}m+8n+4, 8n)$ is balanced, P_2 wins by Theorem 2.3.

This completes the proof of our claim that the configuration in (8.14) is losing on H_1.

8.11.2. *When* $K \geqslant 12$ *and* $K \equiv i \bmod 8$ *for some* $i \in \{0,1,3,4,5,6,7\}$. For $m \in \quad_0$, we focus on

$$w_0 \quad B) = 2^{R+1}m+4,\ w_0\ BC) = 3,\ w_0\ CD) = 2^{R+1}m+\ K \quad 6),\ w_0\ EF) = \ K+1), \tag{8.16}$$

where $R = f\ K+1)$. The first round of the game played on this initial configuration unfolds as follows:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. The following cases are possible:

(a) If $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{4,5, \quad ,K \quad 7\}$, with $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,5,6,7\}$, and $w_1\ BC) = 3$, P_2 removes weight $\ K \quad 6 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+7)+1}2^{R \quad f\ L+7)}m+4,\ w_2\ BC) = 3,$$
$$w_2\ CD) = 2^{f\ L+7)+1}2^{R \quad f\ L+7)}m+L,\ w_2\ EF) = \ L+7),$$

which is of the form (4.6) with $r = 4$, $s = 3$ and $w_2\ EF) = \ L+7) \leqslant K$, so that P_1 loses by our induction hypothesis.

(b) If $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{4,5, \quad ,K \quad 7\}$, with $L \equiv 4 \bmod 8$, and $w_1\ BC) = 3$, P_2 removes weight $\ K+2 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L \quad 1)+1}2^{R \quad f\ L \quad 1)}m+4,\ w_2\ BC) = 3,$$
$$w_2\ CD) = 2^{f\ L \quad 1)+1}2^{R \quad f\ L \quad 1)}m+L,\ w_2\ EF) = \ L \quad 1),$$

which is of the form (4.5) with $r = 4$, $s = 3$ and $w_2\ EF) = \ L \quad 1) \leqslant \ K \quad 8) < K$, so that P_1 loses by our induction hypothesis.

(c) If $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{4,5, \quad ,K \quad 7\}$, with $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,6,7\}$, and $w_1\ BC) = 2$, P_2 removes weight $\ K \quad 5 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+6)+1}2^{R \quad f\ L+6)}m+4,\ w_2\ BC) = 2,$$
$$w_2\ CD) = 2^{f\ L+6)+1}2^{R \quad f\ L+6)}m+L,\ w_2\ EF) = \ L+6),$$

which is of the form (4.6) with $r = 4$ and $s = 2$, and by what we have already proved in §8.10, we conclude that P_1 loses.

(d) If $w_1\ CD) = 2^{R+1}m+L$ for $L \in \{4,5, \quad ,K \quad 7\}$, with $L \equiv j \bmod 8$ for $j \in \{4,5\}$, and $w_1\ BC) = 2$, P_2 removes weight $\ K+3 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L \quad 2)+1}2^{R \quad f\ L \quad 2)}m+4,\ w_2\ BC) = 2,$$
$$w_2\ CD) = 2^{f\ L \quad 2)+1}2^{R \quad f\ L \quad 2)}m+L,\ w_2\ EF) = \ L \quad 2),$$

which is of the form (4.5) with $r = 4$ and $s = 2$, and by what we have already proved in §8.10, we conclude that P_1 loses.

(e) If $w_1\ CD) = 2^{R+1}m + L$ for $L \in \{4, 5, \quad , K \quad 7\}$, with $L \equiv j$ mod 8 for $j \in \{0, 1, 2, 3, 7\}$, and $w_1\ BC) = 1$, P_2 removes weight $\ K \quad 4 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+5)+1} 2^{R\ \ f\ L+5)} m + 4,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L+5)+1} 2^{R\ \ f\ L+5)} m + L,\ w_2\ EF) = \ L + 5),$$

which is of the form (4.6) with $r = 4$ and $s = 1$, and by what we have already proved in §8.9, we conclude that P_1 loses.

(f) If $w_1\ CD) = 2^{R+1}m + L$ for $L \in \{4, 5, \quad , K \quad 7\}$, with $L \equiv j$ mod 8 for $j \in \{4, 5, 6\}$, and $w_1\ BC) = 1$, P_2 removes weight $\ K + 4 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L \quad 3)+1} 2^{R\ \ f\ L \quad 3)} m + 4,\ w_2\ BC) = 1,$$
$$w_2\ CD) = 2^{f\ L \quad 3)+1} 2^{R\ \ f\ L \quad 3)} m + L,\ w_2\ EF) = \ L \quad 3),$$

which is of the form (4.5) with $r = 4$ and $s = 1$, and by what we have already proved in §8.9, we conclude that P_1 loses.

(g) If $w_1\ CD) = 2^{R+1}m + L$ for $L \in \{0, 1, 2, 3\}$, and $w_1\ BC) = t \in \{1, 2, 3\}$, P_2 removes weight $\ K \quad 3 \quad t \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m + L,\ w_2\ EF) = \ 4 + t + L),$$

which is of the form (4.4) with either $r = 0$ (when $L = 0$) or $r = 1$ (when $L = 1$) or $r = 3$ (when $L = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $L = 2$ and $t = 1$), or of the form (4.7) with $r = 2$ and $s \geqslant 2$ (when $L = 2$ and $t \geqslant 2$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses in each of these cases.

(h) If $w_1\ CD) = 2^{R+1}m + L$ for $L \in \{0, 1, \quad , K \quad 7\}$, and $w_1\ BC) = 0$, P_2, after the first round, is left with a galaxy graph, consisting of the edges $\ B$, CD and EF,with $w_1 \quad B) = 2^{R+1}m + 4$, $w_1\ CD) = 2^{R+1}m + L$ and $w_1\ EF) = \ K + 1)$, and P_2 wins by Lemma 4.13.

(i) Finally, if $w_1\ CD) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell_1 \in \{0, 1, \quad , 2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from $\ B$, and a non-negative integer weight from BC, in the first round. The following scenarios are possible:

(a) Suppose $w_1 \quad B) = 2^{R+1}m + \ell$ for some $\ell \in \{0, 1, 2, 3\}$ and $w_1\ BC) = t \in \{1, 2, 3\}$. s long as (i) $\ell \in \{0, 1, 3\}$, (ii) or $\ell = 2$ and $t \in \{2, 3\}$, (iii) or $\ell = 2$, $t = 1$ and $K \equiv i$ mod 8 for some $i \in \{1, 3, 5, 6, 7\}$, P_2 removes weight $\ 7 \quad \ell \quad t)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + \ell,\ w_2\ BC) = t,\ w_2\ CD) = 2^{R+1}m + K \quad 6,\ w_2\ EF) = K + \ell + t \quad 6,$$

which is of the form (4.4) with $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$) or $r = 3$ (when $\ell = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $\ell = 2$, $t = 1$ and $K \equiv i$ mod 8 for some $i \in \{1, 3, 5, 6, 7\}$), or of the form (4.7) with $r = 2$ and $s \geqslant 2$ (when $\ell = 2$ and $t \in \{2, 3\}$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses.

If $\ell = 2$, $t = 1$ and $K \equiv i$ mod 8 for some $i \in \{0, 4\}$, P_2 removes weight 8 from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 2,\ w_2\ BC) = 1,\ w_2\ CD) = 2^{R+1}m + K \quad 6,\ w_2\ EF) = K \quad 7,$$

which is of the form (4.5) with $r = 2$ and $s = 1$, and by what we have already proved in §7.6, we conclude that P_1 loses.

(b) If $w_1 \quad B) = 2^{R+1}m + \ell$ for $\ell \in \{0,1,2,3\}$, and $w_1\ BC) = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges $\quad B$, CD and EF, with $w_1 \quad B) = 2^{R+1}m + \ell$, $w_1\ CD) = 2^{R+1}m + \ K \quad 6)$ and $w_1\ EF) = \ K + 1)$, and she wins by Lemma 4.13.

(c) Finally, if $w_1 \quad B) = 2^{R+1}n + \ell_1$ for some $n < m$ and $\ell_1 \in \{0,1, \quad ,2^{R+1} \quad 1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = K \quad 6$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from the edge EF in the first round, so that $w_1\ EF) = k \leqslant K$. If $k = 0$, P_2 wins by Remark 4.12. If $k \in \{1,2,3,4,K \quad 6\}$, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = \ K \quad 6)$ and $w_1\ EF) = k$ satisfying either $\min\{\ell_1,\ell_2\} \geqslant k$ or $k \in \{\ell_1,\ell_2\}$. Therefore, P_2 wins by Lemma 4.9. If $k \in \{K \quad 3, K \quad 2, K \quad 1, K\}$, P_2 removes $\ K+1 \quad k)$ from $\quad B$ in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 3 + k \quad K,\ w_2\ BC) = 3,\ w_2\ CD) = 2^{R+1}m + K \quad 6,\ w_2\ EF) = k,$$

which is of the form (4.4) with either $r = 0$ (when $k = K \quad 3$) or $r = 1$ (when $k = K \quad 2$) or $r = 3$ (when $k = K$), or of the form (4.7) with $r = 2$ and $s = 3$ (when $k = K \quad 1$). If $k \in \{K \quad 5, K \quad 4\}$, P_2 removes weight 4 from $\quad B$ and weight $\ K \quad k \quad 3)$ from BC in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m,\ w_2\ BC) = k + 6 \quad K,\ w_2\ CD) = 2^{R+1}m + K \quad 6,\ w_2\ EF) = k,$$

which is of the form (4.4) with $r = 0$. By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we conclude that P_1 loses in each of these cases.

If $k \in \{7,8,9,10,12,13, \quad ,K \quad 7\}$ with $k \equiv j \bmod 8$ for $j \in \{0,1,2,4,5,6,7\}$, P_2 removes weight $\ K+1 \quad k)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = 3,\ w_2\ CD) = 2^{R+1}m + \ k \quad 7),\ w_2\ EF) = k, \tag{8.17}$$

which, for $k \geqslant 12$, is of the form (4.6) with $r = 4$ and $s = 3$, and as $w_2\ EF) = k \leqslant K$, P_1 loses by our induction hypothesis. The configuration in (8.17) is of the form (4.4) with either $r = 0$ (when $k = 7$) or $r = 1$ (when $k = 8$) or $r = 3$ (when $k = 10$), or of the form (4.7) with $r = 2$ and $s = 3$ (when $k = 9$). Note that the case of $k = \ K \quad 7)$ is taken care of here. If $k \in \{11,12, \quad ,K \quad 8\}$ with $k \equiv 3 \bmod 8$, P_2 removes weight $\ K \quad 7 \quad k)$ from CD in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = 3,\ w_2\ CD) = 2^{R+1}m + \ k+1),\ w_2\ EF) = k,$$

which is of the form (4.5) with $r = 4$ and $s = 3$, and as $w_2\ EF) = k \leqslant K$, P_1 loses by our induction hypothesis. If $k \in \{5,6\}$, P_2 removes weight $\ K \quad 6)$ from CD, and weight $\ 7 \quad k)$ from BC, in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = k \quad 4,\ w_2\ CD) = 2^{R+1}m,\ w_2\ EF) = k,$$

which is of the form (4.4) with $r = 0$. By what we have already proved in §7.4, §8.5, §8.8 and §7.7, we conclude that P_1 loses in each of these cases.

(iv) Finally, suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of $\quad B$ and CD, so that $w_1\ BC) = t \in \{0,1,2\}$. If $t = 2$ and $K \equiv i \bmod 8$ for $i \in \{0,1,4,5,6,7\}$, P_2 removes weight 1 from EF in the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = 2,\ w_2\ CD) = 2^{R+1}m + \ K \quad 6),\ w_2\ EF) = K,$$

which is of the form (4.6) with $r = 4$ and $s = 2$. If $t = 2$ and $K \equiv 3 \bmod 8$, P_2 removes weight 9 from EF in the second round, so that P_1 is left with

$$w_2 \quad B) = 2^{R+1}m + 4,\ w_2\ BC) = 2,\ w_2\ CD) = 2^{R+1}m + \ K \quad 6),\ w_2\ EF) = \ K \quad 8),$$

which is of the form (4.5) with $r = 4$ and $s = 1$. By what we have already proved in §8.10, we conclude that P_1 loses in each of these cases.

If $t=1$ and $K\equiv i \bmod 8$ for $i\in\{0,1,5,6,7\}$, P_2 removes weight 2 from EF in the second round, leaving P_1 with

$$w_2(AB)=2^{R+1}m+4,\ w_2(BC)=1,\ w_2(CD)=2^{R+1}m+(K-6),\ w_2(EF)=(K-1),$$

which is of the form (4.6) with $r=4$ and $s=1$. If $t=1$ and $K\equiv i \bmod 8$ for $i\in\{3,4\}$, P_2 removes weight 10 from EF in the second round, so that P_1 is left with

$$w_2(AB)=2^{R+1}m+4,\ w_2(BC)=2,\ w_2(CD)=2^{R+1}m+(K-6),\ w_2(EF)=(K-9),$$

which is of the form (4.5) with $r=4$ and $s=1$. By what we have already proved in §8.9, we conclude that P_1 loses.

If $t=0$, P_2 is left with a galaxy graph at the end of the first round, consisting of edges AB, CD and EF, with $w_1(AB)=2^{R+1}m+4$, $w_1(CD)=2^{R+1}m+(K-6)$ and $w_1(EF)=(K+1)$, and she wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.16) is losing, thereby proving our claim that any configuration that is either of the form (4.5) or of the form (4.6), with $r=4$ and $s=3$, is losing.

8.12. **Proof that any configuration of the form** (4.7)**, with** $r=4$ **and** $s\geqslant 4$**, is losing.** The base case corresponding to (4.7) with $r=4$ is obtained by setting $s=4$ and $k=12$, which yields the configuration

$$w_0(AB)=w_0(CD)=16m+4,\ w_0(BC)=4,\ w_0(EF)=12 \tag{8.18}$$

We consider the first round of the game played on this initial configuration:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round (analogously, P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, in the first round).

(a) If $w_1(CD)=16m+\ell$ for some $\ell\in\{0,1,2,3\}$, and $w_1(BC)=t\in\{1,2,3,4\}$, P_2 removes weight $(8-\ell-t)$ from the edge EF in the second round, leaving P_1 with

$$w_2(AB)=16m+4,\ w_2(BC)=t,\ w_2(CD)=16m+\ell,\ w_2(EF)=4+\ell+t,$$

which is of the form (4.4) with either $r=0$ (when $\ell=0$) or $r=1$ (when $\ell=1$) or $r=3$ (when $\ell=3$), or of the form (4.6) with $r=2$ and $s=1$ (when $\ell=2$ and $t=1$), or of the form (4.7) with $r=2$ and $s\geqslant 2$ (when $\ell=2$ and $t\in\{2,3\}$). By what we have proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we conclude that P_1 loses in each of these cases.

(b) If $w_1(CD)=16m+\ell$ for $\ell\in\{0,1,2,3\}$, and $w_1(BC)=0$, P_2, at the end of the first round, is left with a galaxy graph that consists of the edges AB, CD and EF, with $w_1(AB)=16m+4$, $w_1(CD)=16m+\ell$ and $w_1(EF)=12$, and she wins by Lemma 4.13.

(ii) Suppose P_1 removes a positive integer weight from EF in the second round, so that $w_1(EF)=k\leqslant 11$. If $k\in\{8,9,10,11\}$, P_2 removes weight $(12-k)$ from CD in the second round, leaving P_1 with

$$w_2(AB)=16m+4,\ w_2(BC)=4,\ w_2(CD)=16m+(k-8),\ w_2(EF)=k,$$

which is of the form (4.4) with either $r=0$ (when $k=8$) or $r=1$ (when $k=9$) or $r=3$ (when $k=11$), or of the form (4.7) with $r=2$ and $s=4$ (when $k=10$). If $k\in\{5,6,7\}$, P_2 removes weight 4 from CD and weight $(8-k)$ from BC in the second round, so that P_1 is left with

$$w_2(AB)=16m+4,\ w_2(BC)=(k-4),\ w_2(CD)=16m,\ w_2(EF)=k,$$

which is of the form (4.4) with $r=0$. By what we have proved already in §7.4, §8.5, §8.8 and §7.7, we conclude that P_1 loses in each of these cases. If $k\in\{1,2,3,4\}$, the configuration at the end of the first round is of the form stated in Lemma 4.9, with $\ell_1=\ell_2=4$ and $w_1(EF)=k$ satisfying $\min\{\ell_1,\ell_2\}\geqslant k$, so that P_2 wins by Lemma 4.9. If $k=0$, P_2 wins by Remark 4.12.

(iii) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of AB and CD, so that $w_1(BC) = t \in \{0,1,2,3\}$. For $t \in \{1,2,3\}$, P_2 removes weight $(12-t)$ from the edge EF in the second round, so that P_1 is left with

$$w_2(AB) = w_2(CD) = 16m+4 = 4(4m+1),\ w_2(BC) = w_2(EF) = t,$$

which is of the form (4.4) with $r = 0$. By what we have proved in §7.4, we conclude that P_1 loses. For $t = 0$, P_2 removes EF in the second round, and P_1 loses by Theorem 2.3.

This completes the proof of our claim that the configuration in (8.18) is losing on H_1.

Suppose we have proved, for some $K \in \mathbb{N}$ with $K \geqslant 12$, that any configuration on H_1 that is of the form (4.7) with $r = 4$ and $k \leqslant K$, is losing. We consider the first round of the game played on the configuration

$$w_0(AB) = 2^{R+1}m+4,\ w_0(BC) = K-L-3,\ w_0(CD) = 2^{R+1}m+L,\ w_0(EF) = K+1, \tag{8.19}$$

where $R = f(K+1)$, $L \in \{4,5,\ldots,K-7\}$ and $m \in \mathbb{N}_0$:

(i) Suppose P_1 removes a positive integer weight from CD, and a non-negative integer weight from BC, in the first round. The following cases may arise:

(a) If $w_1(CD) = 2^{R+1}m+\ell$ for $\ell \in \{4,5,\ldots,L-1\}$ and $w_1(BC) = t \in \{4,5,\ldots,K-L-3\}$, P_2 removes weight $(K-3-\ell-t)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell+t+4)+1}2^{R-f(\ell+t+4)}m+4,\ w_2(BC) = t,$$
$$w_2(CD) = 2^{f(\ell+t+4)+1}2^{R-f(\ell+t+4)}m+\ell,\ w_2(EF) = \ell+t+4,$$

which is of the form (4.7) with $r = 4$, $s \geqslant 4$ and $w_2(EF) = \ell+t+4 \leqslant K$, so that P_1 loses by our induction hypothesis.

(b) If $w_1(CD) = 2^{R+1}m+\ell$ for $\ell \in \{4,5,\ldots,L-1\}$ with $\ell \equiv 4 \bmod 8$, and $w_1(BC) = 3$, P_2 removes weight $(K+2-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell-1)+1}2^{R-f(\ell-1)}m+4,\ w_2(BC) = 3,$$
$$w_2(CD) = 2^{f(\ell-1)+1}2^{R-f(\ell-1)}m+\ell,\ w_2(EF) = \ell-1,$$

which is of the form (4.5) with $r = 4$ and $s = 3$. If $\ell \equiv i \bmod 8$ for $i \in \{0,1,2,3,5,6,7\}$, P_2 removes weight $(K-6-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell+7)+1}2^{R-f(\ell+7)}m+4,\ w_2(BC) = 3,$$
$$w_2(CD) = 2^{f(\ell+7)+1}2^{R-f(\ell+7)}m+\ell,\ w_2(EF) = \ell+7,$$

which is of the form (4.6) with $r = 4$ and $s = 3$. By what we have already proved in §8.11, we know that P_2 loses in each of these cases.

(c) Suppose $w_1(CD) = 2^{R+1}m+\ell$ for $\ell \in \{4,5,\ldots,L-1\}$ and $w_1(BC) = 2$. If $\ell \equiv i \bmod 8$ for $i \in \{4,5\}$, P_2 removes weight $(K+3-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell-2)+1}2^{R-f(\ell-2)}m+4,\ w_2(BC) = 2,$$
$$w_2(CD) = 2^{f(\ell-2)+1}2^{R-f(\ell-2)}m+\ell,\ w_2(EF) = \ell-2,$$

which is of the form (4.5) with $r = 4$ and $s = 2$. If $\ell \equiv i \bmod 8$ for $i \in \{0,1,2,3,6,7\}$, P_2 removes weight $(K-5-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell+6)+1}2^{R-f(\ell+6)}m+4,\ w_2(BC) = 2,$$
$$w_2(CD) = 2^{f(\ell+6)+1}2^{R-f(\ell+6)}m+\ell,\ w_2(EF) = \ell+6,$$

which is of the form (4.6) with $r = 4$ and $s = 2$. By what we have already proved in §8.10, we know that P_2 loses in each of these cases.

(d) Suppose $w_1(CD) = 2^{R+1}m+\ell$ for some $\ell \in \{4,5,\ldots,L-1\}$ and $w_1(BC) = 1$. If $\ell \equiv i \bmod 8$ for $i \in \{4,5,6\}$, P_2 removes weight $(K+4-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell-3)+1}2^{R-f(\ell-3)}m+4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(\ell-3)+1}2^{R-f(\ell-3)}m+\ell,\ w_2(EF) = \ell-3,$$

which is of the form (4.5) with $r = 4$ and $s = 1$. If $\ell \equiv i \bmod 8$ for $i \in \{0,1,2,3,7\}$, P_2 removes weight $(K-4-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(\ell+5)+1}2^{R-f(\ell+5)}m+4,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(\ell+5)+1}2^{R-f(\ell+5)}m+\ell,\ w_2(EF) = \ell+5,$$

which is of the form (4.6) with $r = 4$ and $s = 1$. By what we have already proved in §8.9, we know that P_2 loses in each of these cases.

(e) If $w_1(CD) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,2,3\}$ and $w_1(BC) = t \in \{1,2,\ldots,K-L-3\}$, P_2 removes weight $(K-3-t-\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{R+1}m+4,\ w_2(BC) = t,\ w_2(CD) = 2^{R+1}m+\ell,\ w_2(EF) = 4+t+\ell,$$

which is of the form (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$) or $r = 3$ (when $\ell = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $\ell = 2$ and $t = 1$), or of the form (4.7) with $r = 2$ and $s \geqslant 2$ (when $\ell = 2$ and $t \in \{2,3,\ldots,K-L-3\}$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_2 loses in each of these cases.

(f) If $w_1(CD) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,\ldots,L-1\}$ and $w_1(BC) = 0$, P_2, after the first round, is left with a galaxy graph consisting of the edges AB, CD and EF, with $w_1(AB) = 2^{R+1}m+4$, $w_1(CD) = 2^{R+1}m+\ell$ and $w_1(EF) = (K+1)$, and she wins by Lemma 4.13.

(g) Finally, if $w_1(CD) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = 4$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(ii) Suppose P_1 removes a positive integer weight from AB, and a non-negative integer weight from BC, in the first round. The following cases may arise:

(a) Suppose $w_1(AB) = 2^{R+1}m+\ell$ for some $\ell \in \{0,1,2,3\}$ and $w_1(BC) = t \in \{1,2,\ldots,K-L-3\}$. As long as (i) either $\ell \in \{0,1,3\}$, (ii) or $\ell = 2$ and $t \in \{2,3,\ldots,K-L-3\}$, (iii) or $\ell = 2$, $t = 1$ and $L \equiv j \bmod 4$ for some $j \in \{0,1,3\}$, P_2 removes weight $(K+1)-(L+t+\ell)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+t+\ell)+1}2^{R-f(L+t+\ell)}m+\ell,\ w_2(BC) = t,$$
$$w_2(CD) = 2^{f(L+t+\ell)+1}2^{R-f(L+t+\ell)}m+L,\ w_2(EF) = L+t+\ell,$$

which is of the form (4.4) with either $r = 0$ (when $\ell = 0$) or $r = 1$ (when $\ell = 1$) or $r = 3$ (when $\ell = 3$), or of the form (4.6) with $r = 2$ and $s = 1$ (when $\ell = 2$, $t = 1$ and $L \equiv j \bmod 4$ for some $j \in \{0,1,3\}$), or of the form (4.7) with $r = 2$ and $s \geqslant 2$ (when $\ell = 2$ and $t \in \{2,3,\ldots,K-L-3\}$). By what we have already proved in §7.4, §8.5, §8.8, §7.6 and §7.7, we know that P_1 loses. If $\ell = 2$, $t = 1$ and $L \equiv 2 \bmod 4$, P_2 removes weight $(K+2-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-1)+1}2^{R-f(L-1)}m+2,\ w_2(BC) = 1,$$
$$w_2(CD) = 2^{f(L-1)+1}2^{R-f(L-1)}m+L,\ w_2(EF) = (L-1),$$

which is of the form (4.5) with $r = 2$ and $s = 1$, and by what we have proved in §7.6, we know that P_1 loses.

(b) If $w_1(AB) = 2^{R+1}m+\ell$ for $\ell \in \{0,1,2,3\}$, and $w_1(BC) = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges AB, CD and EF, where $w_1(AB) = 2^{R+1}m+\ell$, $w_1(CD) = 2^{R+1}m+L$ and $w_1(EF) = (K+1)$, and she wins by Lemma 4.13.

(c) Finally, if $w_1(AB) = 2^{R+1}n+\ell_1$ for some $n < m$ and $\ell_1 \in \{0,1,\ldots,2^{R+1}-1\}$, the configuration after the first round is of the form given by Lemma 4.9 with $m_1 = n$, $m_2 = m$ and $\ell_2 = L$. Since $n < m \implies m_1 \neq m_2$, P_2 wins by Lemma 4.9.

(iii) Suppose P_1 removes a positive integer weight from EF in the first round, so that $w_1(EF) = k \leqslant K$. For $k \in \{K-L+1,\ldots,K\}$, P_2 removes $(K+1-k)$ from CD in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m+4,\ w_2(BC) = (K-L-3),$$
$$w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m+(k+L-K-1),\ w_2(EF) = k,$$

which is of the form (4.7) with $r=4$ and $s \geqslant 4$ when $k \in \{K-L+5,\ldots,K\}$, of the form (4.4) with either $r=3$ (when $k = (K-L+4)$) or $r=1$ (when $k = (K-L+2)$) or $r=0$ (when $k = (K-L+1)$), or of the form (4.7) with $r=2$ and $s \geqslant 4$ when $k = (K-L+3)$. In the first case, since $w_2(EF) = k \leqslant K$, P_1 loses by our induction hypothesis; in each of the latter cases, P_1 loses by what we have already proved in §7.4, §8.5, §8.8 and §7.7.

For $k \in \{5,6,\ldots,K-L\}$, P_2 removes weight L from CD and weight $(K-L+1-k)$ from BC in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(k)+1}2^{R-f(k)}m+4,\ w_2(BC) = (k-4),\ w_2(CD) = 2^{f(k)+1}2^{R-f(k)}m,\ w_2(EF) = k,$$

which is of the form (4.4) with $r=0$, and by what we have already proved in §7.4, we conclude that P_1 loses. For $k \in \{1,2,3,4\}$, the configuration after the first round is of the form mentioned in Lemma 4.9, with $\ell_1 = 4$, $\ell_2 = L$ and $w_1(EF) = k$ satisfying $\min\{\ell_1,\ell_2\} \geqslant k$, so that P_2 wins by Lemma 4.9. If $k=0$, P_2 wins by Remark 4.12.

(iv) Suppose P_1 removes a positive integer weight from BC in the first round, without disturbing the edge-weights of AB and CD, so that $w_1(BC) = t \in \{0,1,\ldots,K-L-4\}$. If $t \in \{4,5,\ldots,K-L-4\}$, P_2 removes weight $(K-L-3-t)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+t+4)+1}2^{R-f(L+t+4)}m+4,\ w_2(BC) = t,$$
$$w_2(CD) = 2^{f(L+t+4)+1}2^{R-f(L+t+4)}m+L,\ w_2(EF) = (L+t+4),$$

which is of the form (4.7) with $r=4$ and $s \geqslant 4$, and as $w_2(EF) = (L+t+4) \leqslant K$, P_1 loses by our induction hypothesis.

If $t=3$ and $L \equiv 4 \bmod 8$, P_2 removes $(K+2-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-1)+1}2^{R-f(L-1)}m+4,\ w_2(BC) = 3,$$
$$w_2(CD) = 2^{f(L-1)+1}2^{R-f(L-1)}m+L,\ w_2(EF) = (L-1),$$

which is of the form (4.5) with $r=4$ and $s=3$. If $t=3$ and $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,5,6,7\}$, P_2 removes $(K-6-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L+7)+1}2^{R-f(L+7)}m+4,\ w_2(BC) = 3,$$
$$w_2(CD) = 2^{f(L+7)+1}2^{R-f(L+7)}m+L,\ w_2(EF) = (L+7),$$

which is of the form (4.6) with $r=4$ and $s=3$. By what we have already proved in §8.11, we conclude that P_1 loses in each of these cases.

If $t=2$ and $L \equiv j \bmod 8$ for $j \in \{4,5\}$, P_2 removes weight $(K+3-L)$ from EF in the second round, leaving P_1 with

$$w_2(AB) = 2^{f(L-2)+1}2^{R-f(L-2)}m+4,\ w_2(BC) = 2,$$

$$w_2\ CD) = 2^{f\ L\ \ 2)+1}2^{R\ \ f\ L\ \ 2)}m + L,\ w_2\ EF) = \ L \quad 2),$$

which is of the form (4.5) with $r = 4$ and $s = 2$. If $t = 2$ and $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,6,7\}$, P_2 removes weight $K \quad 5 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+6)+1}2^{R\ \ f\ L+6)}m + 4,\ w_2\ BC) = 2,$$

$$w_2\ CD) = 2^{f\ L+6)+1}2^{R\ \ f\ L+6)}m + L,\ w_2\ EF) = \ L + 6),$$

which is of the form (4.6) with $r = 4$ and $s = 2$. By what we have already proved in §8.10, we conclude that P_1 loses in each of these cases.

If $t = 1$ and $L \equiv j \bmod 8$ for $j \in \{4,5,6\}$, P_2 removes weight $K + 4 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L\ \ 3)+1}2^{R\ \ f\ L\ \ 3)}m + 4,\ w_2\ BC) = 1,$$

$$w_2\ CD) = 2^{f\ L\ \ 3)+1}2^{R\ \ f\ L\ \ 3)}m + L,\ w_2\ EF) = \ L \quad 3),$$

which is of the form (4.5) with $r = 4$ and $s = 1$. If $t = 1$ and $L \equiv j \bmod 8$ for $j \in \{0,1,2,3,7\}$, P_2 removes weight $K \quad 4 \quad L)$ from EF in the second round, leaving P_1 with

$$w_2 \quad B) = 2^{f\ L+5)+1}2^{R\ \ f\ L+5)}m + 4,\ w_2\ BC) = 1,$$

$$w_2\ CD) = 2^{f\ L+5)+1}2^{R\ \ f\ L+5)}m + L,\ w_2\ EF) = \ L + 5),$$

which is of the form (4.6) with $r = 4$ and $s = 1$. By what we have already proved in §8.9, we conclude that P_1 loses in each of these cases.

If $t = 0$, P_2, at the end of the first round, is left with a galaxy graph consisting of the edges B, CD and EF, where $w_1 \quad B) = 2^{R+1}m + 4$, $w_1\ CD) = 2^{R+1}m + L$ and $w_1\ EF) = \ K + 1)$, and she wins by Lemma 4.13.

This completes the proof of our claim that the configuration in (8.19) is losing, and thereby also completes our inductive proof of the claim that any configuration on H_1 that is of the form (4.7) with $r = 4$, is losing.

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REFERENCES

LBERT, M. H. ND R. J. NOW KOWSKI (2004): "Nim restrictions," *Integers*, 4, G1.

BOUTON, C. L. (1901): "Nim, a game with a complete mathematical theory," *nnals of mathematics*, 3, 35–39.

BROWN, D. E. ND T. K. WILLI MS (2019): "Graph NIM," *Congressus Numerantium*, 233, 151–164.

BRU LDI, R. . (1977): *Introductory combinatorics*, Pearson Education India.

C LKIN, N. J., K. J MES, J. E. J NOSKI, S. LEGGETT, B. RICH RDS, N. SIT R M N, ND S. M. THOM S (2010): "Computing strategies for graphical Nim," in *Proceedings of the Forty-First Southeastern International Conference on Combinatorics, Graph Theory and Computing*, vol. 202, 171–185.

CONW Y, J. H. (2000): *On numbers and games*, K Peters/CRC Press.

DUCHÊNE, E., M. DUFOUR, S. HEUB CH, ND U. L RSSON (2016): "Building nim," *International Journal of Game Theory*, 45, 859–873.

ERICKSON, L. . (2011): *The game of Nim on graphs*, North Dakota State University.

FERGUSON, T. S. (2020): *course in game theory*, World Scientific.

FUKUY M , M. (2003): " Nim game played on graphs," *Theoretical Computer Science*, 304, 387–399.

——— (2003 B): " Nim game played on graphs II," *Theoretical computer science*, 304, 401–419.

G LE, D. (1974): " curious Nim-type game," *The merican Mathematical Monthly*, 81, 876–879.

GRUNDY, P. M. (1939): "Mathematics and games," *Eureka*, 2, 6–8.

GURVICH, V. ND N. B. HO (2015): "Slow *k*-Nim," *arXiv preprint arXiv:1508.05777*.

LI, S.-Y. (1978): "N-person nim and n-person moore's games," *International Journal of Game Theory*, 7, 31–36.

LOW, R. M. ND W. H. CH N (2016): "Notes on the combinatorial game: graph Nim." *Electron. J. Graph Theory ppl.*, 4, 190–205.

LV, X., R. XU, ND X. ZHU (2018): "Greedy Nim k Game," *Journal of Combinatorial Optimization*, 35, 1241–1249.

MOORE, E. H. (1910): " generalization of the game called nim," *nnals of Mathematics*, 11, 93–94.

ROBIN, . (1989): " poisoned chocolate problem, Problem corner," *The Mathematical Gazette*, 73, 341–343.

SCHW RTZ, B. L. (1971): "Some extensions of Nim," *Mathematics Magazine*, 44, 252–257.

SPR GUE, R. (1935): "Über mathematische kampfspiele," *Tohoku Mathematical Journal, First Series*, 41, 438–444.

V N DEN BERGH, M., W. KOSTERS, ND F. SPIEKSM (2022): "Nim variants," *ICG Journal*, 44, 2–17.

WILLI MS, T. K. (2017): *Combinatorial Games on Graphs*, Utah State University.

XU, R. ND X. ZHU (2018): "Bounded Greedy Nim," *Theoretical Computer Science*, 746, 1–5.