SIMULTANEOUS INFERENCE FOR TIME-VARYING MODELS

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A general class of time-varying regression models which cover general linear models as well as time series models is considered. We estimate the regression coefficients by using local linear M-estimation. For these estimators, weak Bahadur representations are obtained and are used to construct simultaneous confidence bands. For practical implementation, we propose a bootstrap based method to circumvent the slow logarithmic convergence of the theoretical simultaneous bands. Our results substantially generalize and unify the treatments for several time-varying regression and auto-regression models. The performance for ARCH and GARCH models is studied in simulations and a few real-life applications of our study are presented through analysis of some popular financial datasets.

Keywords and phrases: Time-varying regression, Time-series models, Generalized linear models, Simultaneous confidence band, Gaussian approximation, Bootstrap

1. Introduction. Time-varying dynamical systems have been studied extensively in the literature of statistics, economics and related fields. For stochastic processes observed over a long time horizon, stationarity is often an over-simplified assumption that ignores systematic deviations of parameters from constancy. For example, in the context of financial datasets, empirical evidence shows that external factors such as war, terrorist attacks, economic crisis, some political event etc. introduce such parameter inconstancy. As Bai [3] points out, 'failure to take into account parameter changes, given their presence, may lead to incorrect policy implications and predictions'. Thus functional estimation of unknown parameter curves using time-varying models has become an important research topic recently. In this paper, we propose a general setting for simultaneous inference of local linear M-estimators in semi-parametric time-varying models. Our formulation is general enough to allow unifying time-varying models from the usual linear regression, generalized regression and several auto-regression type models together. Before discussing our new contributions in this paper, we provide a brief overview of some previous works in these areas.

In the regression context, time-varying models are discussed over the past two decades to describe non-constant relationships between the response and the predictors; see, for instance, Fan and Zhang [19], Fan and Zhang [20], Hoover et al. [27], Huang, Wu and Zhou [28], Lin and Ying [37], Ramsay and Silverman [45], Zhang, Lee and Song [58] among others. Consider the following two regression models

$$\text{Model I: } y_i = x_i^\top \theta_i + e_i, \quad \text{Model II: } y_i = x_i^\top \theta_0 + e_i, \qquad i = 1, \ldots, n,$$

where $x_i \in \mathbb{R}^d$ $(i = 1, \ldots, n)$ are the covariates, $^\top$ is the transpose, θ_0 and $\theta_i = \theta(i/n)$ are the regression coefficients. Here, θ_0 is a constant parameter and $\theta : [0, 1] \to \mathbb{R}^d$ is a smooth function. Estimation of $\theta(\cdot)$ has been considered by Hoover et al. [27], Cai [9]) and Zhou and Wu [63] among others. Hypothesis testing is widely used to choose between model I and model II, see, for instance, Zhang and Wu [59], Zhang and Wu [60], Chow [11], Brown, Durbin and Evans [7], Nabeya and Tanaka [41], Leybourne and McCabe [34], Nyblom [42], Ploberger, Krämer and Kontrus [44], Andrews [2] and Lin and Teräsvirta [35]. Zhou and Wu [63] discussed obtaining simultaneous confidence bands (SCB) in model I, i.e. with additive errors. However their treatment is heavily based on the closed-form solution and it does not extend to processes defined by a more general recursion. Our framework allows us to perform inference on a much larger class of regression settings. Moreover, it can also accommodate generalized linear models as shown in Section 5. Little has been known for time-varying models in this direction previously.

The results from time-varying linear regression can be naturally extended to time-varying AR, MA or ARMA processes. However, such an extension is not obvious for conditional heteroscedastic (CH) models. These are difficult to estimate but also often more useful in analyzing and predicting financial datasets. Since Engle [17] introduced the classical ARCH model and Bollerslev [6] extended it to a more general GARCH model, these have remained primary tools for analyzing and forecasting certain trends for stock market datasets. As the market is vulnerable to frequent changes, non-uniformity across time is a natural phenomenon. The necessity of extending these classical models to a set-up where the parameters can change across time has been pointed out in several references; for example Stărică and Granger [49], Engle and Rangel [18] and Fryzlewicz, Sapatinas and Subba Rao [24]. Towards time-varying parameter models in the CH setting, numerous works discussed the CUSUM-type procedure, for instance, Kim, Cho and Lee [30] for testing change in parameters of GARCH(1,1). Kulperger et al. [33] studied the high moment partial sum process based on residuals and applied it to residual CUSUM tests in GARCH models. Interested readers can find some more changepoint detection results in the context of CH models in James Chu [29], Chen and Gupta [10], Lin et al. [36], Kokoszka et al. [31] or Andreou and Ghysels [1].

Historically in the analysis of financial datasets, the common practice to account for the time-varying nature of the parameter curves was to transfer a stationary tool/method in some ad hoc way. For example, in Mikosch and Stărică [39], the authors analyzed S&P500 data from 1953-1990 and suggested that time-varying parameters are more suitable due to such a long time-horizon. They re-estimated the parameters for every block of 100 sample points and to account for the abrupt fluctuation of the coefficients, they generated re-estimates of parameters for samples of size $100, 200, \ldots$. This treatment suffers from different degree of reliability of the estimators at different parts of the time horizon. There are examples outside the analysis of economic datasets, where similar approach of splitting the time-horizon has been adapted to fit CH type models. For example, in Giacometti et al. [25], the authors analyzed Italian mortality rates from 1960-2003 using an AR(1)-ARCH(1) model and observed abrupt behavior of yearwise coefficients. We capture all these models together and provide significant improvements over such heuristic treatments. For the convenience of the readers, we summarize our contributions in this paper below after a brief literature overview.

A time-varying framework and a pointwise curve estimation using M-estimators for locally stationary ARCH models was provided by Dahlhaus and Subba Rao [15]. Since then, while several pointwise approaches were discussed in the tvARMA and tvARCH case (cf. Dahlhaus and Polonik [13], Dahlhaus and Subba Rao [15], Fryzlewicz, Sapatinas and Subba Rao [24]), pointwise theoretical results for estimation in tvGARCH processes were discussed in Rohan and Ramanathan [48] and Rohan [47] for GARCH(1,1) and GARCH(p,q) models. Specifically, for the ARMA-GARCH type models there has been some attempts for the pointwise inference and thus calls for a natural although significantly more challenging extension to simultaneous inference. On the other hand, coming to the regime of time-varying generalized linear models, even pointwise inference remained largely untouched, almost surprisingly so, since time-varying generalized regression arise very naturally in econometrics and a large number of other scientific fields. For example, autoregressive logistic models are commonly used in conjunction with longitudinal data. For medical research and biology, see de Vries et al. [16], Kowsar et al. [32] etc; for climatology, see Guanche, Mínguez and Méndez [26]; for risk management analysis see Taylor and Yu [51] etc. Time-varying logit models also have wide applications in recommendation systems, environmental economics, public economics, transportation economics etc. We believe this apparent gap of an inferential framework for these models is due to the inherent correlation between possibly endogenous covariates and the possibly non-linear link function.

Our contributions in this paper is multi-directional. We provide a unifying framework that binds linear regression models, generalized regression models and many popularly used auto-regressive models including CH type processes where simultaneous inference for each of these classes are probably important contributions on their own. In fact, our framework is significantly different from that introduced in [15] and draws its general motivation from the unification perspective. Through an introduction of a smoothness class in subsection 2.4 and Assumptions 2.1 or Assumptions 2.2, we provided our results for this new framework. Also, it is important to note that these assumptions are written to accommodate this vast generality and can be simplified without any essential difficulty for the subclasses. To the best of our knowledge, this is the first such attempt in unifying time-varying models from diverse fields in one single thread. Even without the particular contribution of constructing simultaneous inference or more generally the theme of time-varying models, the fact that our framework binds such a large class of models can be exploited in other areas of statistics literature. As an instance, generalizing from our smoothness class here defined using negative log-likelihood, one of the authors is currently exploiting this framework in the light of density power divergence measures (Basu et al. [4]) to extend robust statistics literature towards time-series models for both time-constant and time-varying parameters.

We use weak Bahadur-type representations, a Gaussian approximation theorem from Zhou and Wu [62] and extreme value Gaussian theory to obtain SCBs for contrasts of the parameter curves. These intervals provide a generalization from testing parameter constancy to testing any particular parametric form such as linear, quadratic, exponential etc. A very general recursion model (cf. (2.1)) is considered and asymptotic results for a local linear M-estimator are provided. To deal with bias expansions, we use the theory about derivative processes which was recently formalized in Dahlhaus, Richter and Wu [14].

It is important to highlight the additional technical challenges involved here because of the vast generality of the model from the simplistic linear model in [63] and how it is solved with novel techniques. In general, Bahadur representations are important for the asymptotic analysis of estimators by approximating them by linear forms. The Bahadur-type representation obtained in subsection 3.2 may be of independent interest due to its general set-up and can itself be thought as a new contribution to the literature of locally stationary processes. Moreover, the technical tools to obtain this Bahadur representation result heavily depends on a series of lemmas (cf. Lemma 7.1-Lemma 7.12) that derives some concentration inequalities and chaining results. These carefully exploit the local stationarity and other smoothness class assumptions and thus is significantly different from the much simpler tools used in [63].

A limitation for the theoretical confidence intervals reported in Section 3 is that it only covers $(b_n, 1-b_n)$ fraction of the entire time-spectrum where b_n denotes the bandwidth for estimation. Note that, these apparently incomplete confidence bands are not new in the literature and it appears in Wu and Zhao [56] or [63] for much simpler models such as

$$y_i = \mu(i/n) + e_i \quad \text{or} \quad y_i = x_i^\intercal \theta(i/n) + e_i$$

respectively. The authors therein made the intervals asymptotically comprehensive by choosing $b_n \to 0$. However, for practical implementation often cross-validated or other optimally chosen bandwidth turns out to be rather large, for example, the same for the linear model case as reported in [63] turns out to be $b_n = 0.25$. Constructing a confidence band for such a high bandwidth covering only $(b_n, 1-b_n)$ fraction of the time-spectrum would simply render too restrictive and practically meaningless. This problem is much more severe for the recursively defined CH type models. Even for much easily estimable time-constant cases the GARCH type of processes we usually require large bandwidth to ensure reasonable estimation quality. It is difficult to go beyond $(b_n, 1-b_n)$ for the theoretical bands since the Gaussian extreme value theory used for the same do not allow us to consider the two ends of the time-spectrum. However, we solve this important issue by observing that our bootstrap step reported in subsection 4.4 uses a Gaussian approximation (cf. Theorem 3.3) that works for all the partial sums of a mean-0 process. This allows for extreme small or large values of time, i.e. $t \in (0, b_n) \cup (1-b_n, 1)$ and thus it can be extended to make the SCBs truly comprehensive. We discuss this boundary consideration in Section 4.

The assumptions in this paper while building the geberal framework are intentionally kept general to suit a large class of models. Later we also simplified some of these for specific prominent subclasses in Section 5. From an application point of view, the tvGARCH processes are probably the most important and sophisticated subclass and it has almost become standard to analyze log-return data using small order GARCH because of its superlative forecasting ability. It was important for us to explore optimality of the conditions present in the literature for tvGARCH processes. Using some matrix arguments and new tools exploiting the recursive equation for GARCH processes we obtained that the existence of $4+\delta$ moments of the error process for some small $\delta > 0$ suffices for the construction of the simultaneous bands which significantly improves the 8 th moment assumption for usually simpler pointwise inference as presented in [47] or [48]. Such a significant relaxation of course comes at the cost of positing much more refined assumptions (cf. Assumption 7.16, 7.17) which we postpone to appendix for the sake of clear exposition.

The rest of the article is organized as follows. In Section 2, we introduce our framework, the functional dependence measure, the assumptions and the M-estimators of the parameter curves. Section 3 consists of the results about the weak Bahadur representation and the SCBs of the related contrasts. Section 4 is dedicated to practical issues which arise when using the SCBs like estimation of the dispersion matrix of the estimator, bandwidth selection and a wild Bootstrap procedure to overcome the slow logarithmic convergence from the theoretical SCB. Moreover, this section also covers the boundary consideration to extend the theoretical but approximate SCB to the entirety of time-spectrum. We discuss some examples to show the general applicability of our framework in Section 5. Some summarized simulation studies and real data applications can be found in Section 6. We defer all the proofs to supplement.

2. Model assumptions and estimators.

2.1. *The model.* For some known family of real-valued (possibly stochastic) functions F_i, we consider the model with time-varying parameter curve

$$Y_i = F_i(X_i, \theta(i/n)), \quad i = 1, \dots, n, \tag{2.1}$$

where n is the number of observations, $X_i = (X_{ij})_{j\in\mathbb{N}}$ and Y_i represent a possibly infinite-dimensional covariate process and the real-valued response process respectively. Here $\theta : [0,1] \to \Theta \subset \mathbb{R}^{d_\Theta}$ is a time varying parameter curve. To cover important time series models, we assume that not X_i itself but some truncated version $X_i^c = (X_{ij}^c)_{j\in\mathbb{N}}$ is observed. Let ζ_i, $i \in \mathbb{Z}$ be independent and identically distributed random variables and $\mathcal{F}_i := (\dots, \zeta_{i-1}, \zeta_i)$. We assume the following form for Y_i and X_i

$$X_i = G_i(\mathcal{F}_i), \qquad Y_i = H_i(\mathcal{F}_i), \quad i = 1, \dots, n, \tag{2.2}$$

where $G_i(\cdot) = (G_{ij}(\cdot))_{j\in\mathbb{N}}$ and $H_i(\cdot)$ are measurable functions.

It is worth noting that we do not necessarily need the representation (2.1) as it is only needed in an optional condition (2.13). Some more general formulations may still fit in the setting of this paper. There are some important special cases of (2.1):

(a) Time-varying time series models: Assume that, $(\varepsilon_i)_{i\in\mathbb{Z}}$ are i.i.d., choose $\zeta_i = \varepsilon_i$ and $X_i = (Y_{i-1}, Y_{i-2}, \dots)$. Then (2.1) translates to

$$Y_i = F((Y_{i-1}, Y_{i-2}, \dots), \theta(i/n), \varepsilon_i),$$

which for instance covers tvARMA, tvARCH, tvGARCH processes. In this context, since only $Y_1, \dots, Y_n$ are observed, one usually has $X_i^c = (Y_{i-1}, \dots, Y_1, 0, 0, \dots)$.

(b) The generalized linear model: By using $F_i(x, \theta) = g_i(x^\top \theta)$, where $g_i : \mathbb{R} \to \mathbb{R}$ serves as a (probably stochastic) link function, (2.1) has the form

$$Y_i = g_i(X_i^\top \theta(i/n)).$$

An important example is logistic regression which is assumed to be time-varying in the following sense:

$$Y_i \sim Bin(m, \pi_i), \quad \log\big(\frac{\pi_i}{1-\pi_i}\big) = X_i^\top \theta(i/n),$$

where X_i could possibly be lagged values of Y_i as well.

In either case, our goal is to estimate $\theta(\cdot)$ from the observations $Z_i^c = (Y_i, X_i^c)$, $i = 1, \dots, n$.

2.2. *The estimator.* In this paper, we focus on local M-estimation: Let $K(\cdot) \in \mathcal{K}$, where $\mathcal{K}$ is the family of non-negative symmetric kernels with support $[-1, 1]$ which are continuously differentiable on $[-1, 1]$ such that $\int_{-1}^{1} |K'(u)|^2 du > 0$. Let $\ell(z, \theta)$ be an objective function. A usual choice is the negative log conditional (Gaussian) likelihood of the model which leads to a minimum distance estimator. Define the local linear likelihood function

$$L^c_{n,b_n}(t, \theta, \theta') := (nb_n)^{-1} \sum_{i=1}^{n} K_{b_n}(t - i/n)\ell(Z^c_i, \theta + \theta'(i/n - t)), \tag{2.3}$$

where $K_{b_n}(\cdot) := K(\cdot/b_n)$. Let $\Theta' := [-R, R]^k$ with some $R > 0$. A local linear estimator of $\theta(t)$, $\theta'(t)$ is given by

$$(\hat{\theta}_{b_n}(t), \widehat{\theta'}_{b_n}(t)) = \underset{(\theta,\theta') \in \Theta \times \Theta'}{\operatorname{argmin}} L^c_{n,b_n}(t, \theta, \theta'), \qquad t \in [0, 1]. \tag{2.4}$$

In Examples 5.1 and 5.2, we discuss applications and choices of ℓ for general recursively defined locally stationary time series models and tvGARCH processes. In Example 5.3, we consider a time-varying logistic regression model with a Binomial likelihood function ℓ.

2.3. *The functional dependence measure.* To state the structure of dependence we use throughout the paper, we introduce a functional dependence measure on the underlying process using the idea of coupling as done in Wu [53]. Assume that a stationary process Z_i has mean 0, $Z_i \in \mathcal{L}_q, q > 0$ and it admits the causal representation

$$Z_i = J(\zeta_i, \zeta_{i-1}, \ldots). \tag{2.5}$$

Suppose that $(\zeta^*_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\zeta_i)_{i\in\mathbb{Z}}$. For some random variable Z, let $\|Z\|_q := (\mathbb{E}|Z|^q)^{1/q}$ denote the $\mathcal{L}_q$-norm of Z. For $j \geq 0$, define the functional dependence measure

$$\delta^Z_q(i) = \|Z_i - Z^*_i\|_q, \tag{2.6}$$

where $\mathcal{F}^*_i$ is a coupled version of $\mathcal{F}_i$ with ζ_0 in $\mathcal{F}_i$ replaced by ζ^*_0,

$$\mathcal{F}^* = (\zeta_i, \zeta_{i-1}, \cdots, \zeta_1, \zeta^*_0, \zeta_{-1}, \zeta_{-2}, \cdots), \tag{2.7}$$

and $Z^*_i = J(\mathcal{F}^*_i)$. Note that $\delta^Z_q(i)$ measures the dependence of Z_i on ζ_0 in terms of the qth moment. The tail cumulative dependence measure $\Delta^Z_q(j)$ for $j \geq 0$ is defined as

$$\Delta^Z_q(j) = \sum_{i=j}^{\infty} \delta^Z_q(i). \tag{2.8}$$

2.4. *The class $\mathcal{H}(M_y, M_x, \chi, \bar{C})$.* To prove uniform convergence of L^c_{n,b_n} and its derivatives w.r.t. θ, we require ℓ to be Lipschitz continuous in direction of θ and to grow at most polynomially in direction of $z = (y, x)$, where the degree is measured by integers $M_y, M_x \geq 1$. We will therefore ask ℓ and its derivatives to be in the class $\mathcal{H}(M_y, M_x, \chi, \bar{C})$ which is defined as follows: Let $\chi = (\chi_i)_{i=1,2,...}$ be a sequence of nonnegative real numbers with $|\chi|_1 := \sum_{i=1}^{\infty} \chi_i < \infty$, and $\bar{C} > 0$ be some constant. Define $|x|_{\chi,1} := \sum_{i=1}^{\infty} \chi_i |x_i|$. Put $\hat{\chi} = (1, \chi)$, and for nonnegative integers d_x, d_y, define the 'polynomial rest'

$$R_{d_y,d_x}(z) := \sum_{\substack{k=0 \\ k+l \leq \max\{d_x,d_y\}}}^{d_y} \sum_{l=0}^{d_x} |y|^k |x|^l_{\chi,1}.$$

A function $g : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \Theta \to \mathbb{R}$ is in $\mathcal{H}(M_y, M_x, \chi, \bar{C})$ if $\sup_{\theta \in \Theta} |g(0, \theta)| \leq \bar{C}$,

$$\sup_z \sup_{\theta \neq \theta'} \frac{|g(z, \theta) - g(z, \theta')|}{|\theta - \theta'|_1 R_{M_y,M_x}(z)} \leq \bar{C}$$

and

$$\sup_\theta \sup_{z \neq z'} \frac{|g(z, \theta) - g(z', \theta)|}{|z - z'|_{\hat{\chi},1} \cdot \{R_{M_y-1,M_x-1}(z) + R_{M_y-1,M_x-1}(z')\}} \leq \bar{C}.$$

If g is vector- or matrix-valued, $g \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ means that every component of g is in $\mathcal{H}(M_y, M_x, \chi, \bar{C})$. In Section 5, we will see that a large class of log Gaussian likelihoods and the usual logistic regression likelihood belongs to $\mathcal{H}(M_y, M_x, \chi, \bar{C})$. In case of time series it often holds that $M = M_x = M_y$, which allows to use a simplified version $R_{M_y,M_x}(z) = 1 + |z|^M_{\hat{\chi},1}$.

2.5. *Assumptions.* In this paper, we prove weak Bahadur representations and construct simultaneous confidence bands for $\hat{\theta}_{b_n}(\cdot)$ and $\widehat{\theta'_{b_n}}(\cdot)$. Clearly, more smoothness assumptions on $\theta(\cdot)$ and ℓ are needed to prove results for the latter one which is postponed to Assumption 2.2.

In the following, we will assume the existence of measurable functions H, G such that $\tilde{Y}_i(t) = H(t, \mathcal{F}_i) \in \mathbb{R}$ and $\tilde{X}_i(t) = G(t, \mathcal{F}_i) \in \mathbb{R}^{\mathbb{N}}$ are well-defined for all $t \in [0, 1]$. These processes will serve as stationary approximations of Y_i, X_i if $|i/n - t| \ll 1$. For brevity, define $\tilde{Z}_i(t) := (\tilde{Y}_i(t), \tilde{X}_i(t)^\intercal)^\intercal$ and $Z_i := (Y_i, X_i^\intercal)^\intercal$. The constant $r \geq 2$ in the following assumption is connected to the number of moments that are assumed for Z_i (cf. (A5) and (A7)), while $\gamma > 1$ is a measure of decay of the dependence which is present in the model.

ASSUMPTION 2.1. *Suppose that for some $r \geq 2$ and some $\gamma > 1$,*

(A1) (Smoothness in θ-direction) ℓ is twice continuously differentiable w.r.t. θ. It holds that $\ell, \nabla_\theta \ell, \nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ for some $M_y, M_x \geq 1$, $\bar{C} > 0$ and $\chi = (\chi_i)_{i=1,2,\ldots}$ with $\chi_i = O(i^{-(1+\gamma)})$.

(A2) (Assumptions on unknown parameter curve) Θ is compact and for all $t \in [0,1]$, $\theta(t)$ lies in the interior of Θ. Each component of $\theta(\cdot)$ is in $C^3[0,1]$.

(A3) (Correct model specification) For all $t \in [0,1]$, the function $\theta \mapsto L(t,\theta) := \mathbb{E}\ell(\tilde{Z}_0(t), \theta)$ is uniquely minimized by $\theta(t)$.

(A4) The eigenvalues of the matrices

$$V(t) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)), \tag{2.9}$$

$$I(t) = \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))^{\mathsf{T}}], \tag{2.10}$$

$$\Lambda(t) = \sum_{j \in \mathbb{Z}} \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) \cdot \nabla_\theta \ell(\tilde{Z}_j(t), \theta(t))^{\mathsf{T}}], \tag{2.11}$$

are bounded from below by some $\lambda_0 > 0$, uniformly in t.

(A5) (Stationary approximation) Let $M = \max\{M_x, M_y\}$. There exist $C_A, C_B, D > 0$ such that for all $n \in \mathbb{N}$, $i = 1, \ldots, n$, $t, t' \in [0,1]$, $j \in \mathbb{N}$:

$$\max\{\|Y_i\|_{rM},\ \|\tilde{Y}_0(t)\|_{rM},\ \|X_{ij}\|_{rM},\ \|\tilde{X}_{0j}(t)\|_{rM}\} \leq D,$$

and

$$\|X_{ij} - \tilde{X}_{ij}(i/n)\|_{rM} \leq C_A n^{-1}, \qquad \|\tilde{X}_{0j}(t) - \tilde{X}_{0j}(t')\|_{rM} \leq C_B |t - t'|,$$

and <u>either</u>

$$\|Y_i - \tilde{Y}_i(i/n)\|_{rM} \leq C_A n^{-1}, \qquad \|\tilde{Y}_0(t) - \tilde{Y}_0(t')\|_{rM} \leq C_B |t - t'| \tag{2.12}$$

<u>or</u> (with χ from (A1))

$$\sup_{x \neq x'} \frac{\|F_i(x,\theta) - F_i(x',\theta)\|_{M_y}}{|x - x'|_{\chi,1}} < \infty. \tag{2.13}$$

(A6) (Negligibility of the truncation) For all i, j: $|X_{ij}^c| \leq |X_{ij}|$. For $1 \leq j \leq i$, $X_{ij} = X_{ij}^c$.

(A7) (Weak dependence) It holds that $\sup_{t\in[0,1]} \delta_{rM}^{\tilde{X}(t)}(k) = O(k^{-(1+\gamma)})$ and <u>either</u> (2.13) <u>or</u> $\sup_{t\in[0,1]} \delta_{rM}^{\tilde{Y}(t)}(k) = O(k^{-(1+\gamma)})$ holds.

Note that (A2), (A3) and (A4) are typical assumptions in M-estimation theory to guarantee convergence of the estimator towards the correct parameter and to use Taylor expansions and bias expansions. The condition on L in (A3) directly implies $0 = \nabla_\theta L(t, \theta(t)) = \mathbb{E}\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))$ under the imposed smoothness conditions, which will be used in the proofs. In many special cases in time series analysis (cf. Example 5.1), it may even occur that $\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))$ is a martingale difference sequence or at least an uncorrelated sequence. In these cases, $\Lambda(t) = I(t)$ such that the verification of (A4) is simplified.

Asking the objective function ℓ to be twice continuously differentiable w.r.t. θ as done in (A1) is a typical condition and is needed to use Taylor expansions. We additionally ask ℓ and its derivatives w.r.t. θ to be in $\mathcal{H}(M_y, M_x, \chi, \tilde{C})$. This is exploited in two ways: It allows quantification of the order of dependence of $\ell(Y_i, X_i, \theta)$ based on the dependence of X_i, Y_i, and it allows to deal with local stationarity by replacing X_i, Y_i by its stationary counterparts. In this context, we especially need a decay condition on the coefficients x_i which appear in ℓ. This decay is quantified by the sequence $\chi = (\chi_i)_{i\in\mathbb{N}}$. We use this rate to show that the observed truncated values X_i^c are negligible compared to X_i and that the overall dependence of $\ell(Y_i, X_i, \theta)$ has the same order as the original sequences Y_i, X_i (cf. (A7)). Lastly, condition (A1) implicitly implies continuity of the matrices appearing in (A4) such that it is enough to show pointwise positive definiteness.

To eliminate bias terms, we state (A5) which asks for smoothness of the processes X_i, Y_i in time direction and the existence of a stationary approximation. Here we consider two different cases. The case (2.13) is dedicated to general linear models which may have discretely distributed observations Y_i and thus would not fulfill a condition like (2.12) for $rM \geq 2$. To prove central limits theorems and to use strong Gaussian approximations, we need a weak dependence assumption which is given in (A7). Let us emphasize the fact that all conditions besides (A5) are formulated for the stationary approximation $\tilde{Z}_i(t) = (\tilde{Y}_i(t), \tilde{X}_i(t))$ which in general allows easier verification and the possibility to use earlier results obtained for stationary settings.

To prove a typical second-order bias decomposition for $\widehat{\theta}_{b_n}'(t)$, we need that the stationary approximations $\tilde{Z}_i(t)$ are differentiable w.r.t. t. The concept of derivative processes in the context of locally stationary processes was originally introduced in Dahlhaus [12] and Dahlhaus and Subba Rao [15] and formalized in Dahlhaus, Richter and Wu [14] especially for processes with Markov structure.

ASSUMPTION 2.2 (Differentiability assumptions). *Suppose that there exist* $M_y', M_x' \geq 2$

such that $M' := \max\{M'_x, M'_y\}$ *fufills* $M' \le rM$ *and*

(B1) $\theta(\cdot) \in C^4[0,1]$.

(B2) $\nabla^2_\theta \ell(z,\theta)$ *is continuously differentiable. It holds that* $\nabla^3_\theta \ell \in \mathcal{H}(M'_y, M'_x, \chi, \bar{C})$, *and for all* $l \in \mathbb{N}_0$, $\partial_{z_l} \nabla^2_\theta \ell \in \mathcal{H}(M'_y - 1, M'_x - 1, \chi', \bar{C}\hat{\chi}_l)$ *with some absolutely summable sequence* $\chi' = (\chi'_i)_{i=1,2,\dots}$.

(B3) $t \mapsto \tilde{Z}_0(t)$ *is continuously differentiable and* $\sup_{t\in[0,1]} \sup_{j\in\mathbb{N}_0} \|\partial_t \tilde{Z}_{0j}(t)\|_{M'} \le D$,

$$\sup_{j\in\mathbb{N}_0} \sup_{t\ne t'} \frac{\|\partial_t \tilde{Z}_{0j}(t) - \partial_t \tilde{Z}_{0j}(t')\|_{M'}}{|t-t'|} \le C_B.$$

Note that the condition $\partial_{x_l} \nabla^2_\theta \ell \in \mathcal{H}(M'_y, M'_x, \chi', \bar{C}\chi_l)$ asks $\nabla^2_\theta \ell$ to be dependent on x_l with a factor of at most χ_l which is a stronger condition than the corresponding condition on $\nabla^2_\theta \ell$ in (A1).

A slightly different set of assumptions (Assumption 7.16, 7.17) which is specifically designed for conditional heteroscedastic models, leading to weaker moment assumptions, is postponed to the appendix. All theoretical results in this paper also hold under this set of assumptions.

3. Main results.

3.1. *Consistency and asymptotic normality.* For $l \ge 0$, define

$$\mu_{K,l} := \int K(x)x^l dx, \qquad \sigma^2_{K,l} := \int K(x)^2 x^l dx.$$

Under weaker assumptions than those needed for the proof of SCBs, we can obtain pointwise consistency and asymptotic normality of the estimators $\hat{\theta}_{b_n}$ and $\widehat{\theta}'_{b_n}$. For matrices A, B, let $A \otimes B$ denote the Kronecker product and

$$A^{\otimes k} = A \otimes \ldots \otimes A \tag{3.1}$$

denote the k-fold Kronecker product.

THEOREM 3.1. *Fix* $t \in (0,1)$. *Let Assumption 2.1 hold with* $r = 2$. *Assume that* $nb_n \to \infty$, $b_n \to 0$.

(i) (Consistency) It holds that $\hat{\theta}_{b_n}(t) - \theta(t) = o_{\mathbb{P}}(1)$.
If additionally $nb_n^3 \to \infty$, *it holds that* $\widehat{\theta}'_{b_n}(t) - \theta'(t) = o_{\mathbb{P}}(1)$.

Assume that $\sup_{j\in\mathbb{N}_0}\sup_{t\in[0,1]}\|Z_{0j}(t)\|_{(2+a)M} < \infty$ *for some* $a > 0$.

(ii) *If* $nb_n^7 \to 0$, *then*

$$\sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t) - b_n^2\frac{\mu_{K,2}}{2}\theta''(t)) \xrightarrow{d} N(0, \sigma_{K,0}^2\cdot V(t)^{-1}I(t)V(t)^{-1}). \tag{3.2}$$

(iii) *If additionally, Assumption 2.2 is fulfilled and* $nb_n^9 \to 0$, *then*

$$\begin{pmatrix}\sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t) - b_n^2\frac{\mu_{K,2}}{2}\theta''(t))\\ \sqrt{nb_n^3}(\widehat{\theta}'_{b_n}(t) - \theta'(t) - b_n^2\frac{\mu_{K,4}}{2\mu_{K,2}}bias(t))\end{pmatrix}$$
$$\xrightarrow{d} N\Big(0, \sigma_{K,0}^2\begin{pmatrix}1 & 0\\ 0 & \mu_{K,2}^{-2}\end{pmatrix}\otimes\{V(t)^{-1}I(t)V(t)^{-1}\}\Big), \tag{3.3}$$

where $bias(t) = \frac{1}{3}\theta^{(3)}(t) + V(t)^{-1}\mathbb{E}[\partial_t\nabla_\theta^2\ell(\tilde{Z}_0(t),\theta(t))]\theta''(t)$.

The results hold true if instead of Assumption 2.1, 2.2, Assumption 7.16, 7.17 with some $r > 2$ is assumed.

Remark The condition $\sup_{j\in\mathbb{N}_0}\|\tilde{Z}_{0j}(t)\|_{(2+a)M} < \infty$ is needed to prove a Lindeberg-type condition. As pointed out in the proof of Theorem 2.9 in Dahlhaus, Richter and Wu [14], it can be dropped if instead $\sup_{j\in\mathbb{N}_0}\|\sup_{t\in[0,1]}|\tilde{Z}_{0j}(t)|\|_{2M} < \infty$ is assumed.

About local constant estimation If instead of (2.3) and (2.4), a local constant estimation via

$$L_{n,b_n,const}^c(t,\theta) := (nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n - t)\ell(Z_i^c,\theta)$$

and $\hat{\theta}_{b_n,const}(t) = \operatorname{argmin}_{\theta\in\Theta} L_{n,b_n}^c(t,\theta)$ is used, one needs more smoothness assumptions on the underlying process to obtain a similar result as in (3.2). If for instance twice differentiability of $t \mapsto \tilde{Z}_0(t)$ is assumed, one obtains

$$\sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t) - b_n^2\frac{\mu_{K,2}}{2}V(t)^{-1}\mathbb{E}[\partial_t^2\nabla_\theta\ell(\tilde{Z}_0(t),\theta)\big|_{\theta=\theta(t)}])$$
$$\xrightarrow{d} N(0, \sigma_{K,0}^2\cdot V(t)^{-1}I(t)V(t)^{-1}).$$

Note that the bias term changes significantly.

3.2. *A weak Bahadur representation for* $\hat{\theta}_{b_n}$, $\widehat{\theta}'_{b_n}$. In the following, we obtain a weak Bahadur representation of $\hat{\theta}_{b_n}$ and $\widehat{\theta}'_{b_n}$ which will be used to construct simultaneous confidence bands. The first part of Theorem 3.2(i) shows that $\hat{\theta}_{b_n}(t) - \theta(t)$ can be approximated by the expression $V(t)^{-1}\nabla_\theta L^c_{n,b_n}(t,\theta(t),\theta'(t))$ as expected due to a standard Taylor argument. The second part of Theorem 3.2(i) deals with approximating this term by a weighted sum of t-free terms, namely

$$(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n-t)h_i, \qquad h_i := \nabla_\theta \ell(\tilde{Z}_i(i/n),\theta(i/n)),$$

which is necessary to apply some earlier results from Zhou and Wu [63]. Similar results are obtained for $\widehat{\theta}'_{b_n}$ in Theorem 3.2(ii). Let $\mathcal{T}_n := [b_n, 1-b_n]$. For some vector or matrix x, let $|x| := |x|_2$ denote its Euclidean or Frobenius norm, respectively.

THEOREM 3.2 (Weak Bahadur representation of $\hat{\theta}_{b_n}, \widehat{\theta}'_{b_n}$). *Let $\beta_n = (nb_n)^{-1/2}b_n^{-1/2}\log(n)^{1/2}$ and put*

$$\tau_n^{(j)} = (\beta_n+b_n)((nb_n)^{-1/2}\log(n)+b_n^{1+j}), \qquad j=1,2.$$

Let Assumption 2.1 be fulfilled with some $r>2$.

(i) It holds that

$$\sup_{t\in\mathcal{T}_n}\Big|V(t)\cdot\big\{\hat{\theta}_{b_n}(t)-\theta(t)\big\}-\nabla_\theta L^c_{n,b_n}(t,\theta(t),\theta'(t))\Big| = O_{\mathbb{P}}(\tau_n^{(1)}), \tag{3.4}$$

$$\sup_{t\in\mathcal{T}_n}\big|\nabla_\theta L^c_{n,b_n}(t,\theta(t),\theta'(t)) - b_n^2\frac{\mu_{K,2}}{2}V(t)\theta''(t) - (nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n-t)h_i\big| = O_{\mathbb{P}}(\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}). \tag{3.5}$$

(ii) If additionally Assumption 2.2 is fulfilled, then

$$\sup_{t\in\mathcal{T}_n}\Big|\mu_{K,2}V(t)\cdot b_n\big\{\widehat{\theta}'_{b_n}(t)-\theta'(t)\big\} - b_n^{-1}\nabla_{\theta'}L^c_{n,b_n}(t,\theta(t),\theta'(t))\Big| = O_{\mathbb{P}}(\tau_n^{(2)}), \tag{3.6}$$

$$\sup_{t\in\mathcal{T}_n}\big|b_n^{-1}\nabla_{\theta'}L^c_{n,b_n}(t,\theta(t),\theta'(t)) - b_n^3\frac{\mu_{K,4}}{2}V(t)\,bias(t) - (nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n-t)\frac{(i/n-t)}{b_n}h_i\big| = O_{\mathbb{P}}(\beta_n b_n^2 + b_n^4 + (nb_n)^{-1}). \tag{3.7}$$

Remark The results hold true if instead of Assumption 2.1, 2.2, Assumption 7.16, 7.17 is assumed.

3.3. *Simultaneous confidence bands for $\hat{\theta}_{b_n}$, $\widehat{\theta}'_{b_n}$.* Based on the weak Bahadur result, we use results from Wu and Zhou [57] to obtain a Gaussian analogue of

$$\frac{1}{nb_n}\sum_{i=1}^{n} K_{b_n}(t-i/n)C^{\mathsf{T}}V(t)^{-1}\nabla_\theta \ell(\tilde{Z}_i(i/n),\theta(i/n)) =: \frac{1}{nb_n}\sum_{i=1}^{n} K_{b_n}(t-i/n)\tilde{h}_i(i/n)$$

for some $C \in \mathbb{R}^{s\times k}$.

For the following results, let us assume that there exists some measurable function $\tilde{H}(\cdot,\cdot)$ such that for each $t \in [0,1]$, $\tilde{h}_i(t) = \tilde{H}(t,\mathcal{F}_i) \in \mathbb{R}^s$ is well-defined. Put $S_{\tilde{h}}(i) := \sum_{j=1}^{i} \tilde{h}_j(j/n)$. For a positive semidefinite matrix A with eigendecomposition $A = QDQ^{\mathsf{T}}$, where Q is orthonormal and D is a diagonal matrix, define $A^{1/2} = QD^{1/2}Q^{\mathsf{T}}$, where $D^{1/2}$ is the elementwise root of D.

THEOREM 3.3 (Theorem 1 and Corollary 2 from Wu and Zhou [57]). *Assume that for each component $j = 1,\dots,s$:*

(a) $\sup_{t\in[0,1]} \|\tilde{h}_0(t)_j\|_{2+\varsigma} < \infty$,
(b) $\sup_{t\neq t'\in[0,1]} \|\tilde{h}_0(t)_j - \tilde{h}_0(t')_j\|_2/|t-t'| < \infty$,
(c) $\sup_{t\in[0,1]} \delta_{2+\varsigma}^{\tilde{h}(t)_j}(k) = O(k^{-(\gamma+1)})$ *with some* $\gamma \geq 1$.

for some $\varsigma \leq 2$. Then on a richer probability space, there are i.i.d. $V_1, V_2,\dots \sim N(0, I_{s\times s})$ and a process $S^0_{\tilde{h}}(i) = \sum_{j=1}^{i}\Sigma_{\tilde{h}}(j/n)V_j$ such that $(S_{\tilde{h}}(i))_{i=1,\dots,n} \overset{d}{=} (S^0_{\tilde{h}}(i))_{i=1,\dots,n}$ and

$$\max_{i=1,\dots,n} |S_{\tilde{h}}(i) - S^0_{\tilde{h}}(i)| = O_{\mathbb{P}}(\pi_n).$$

where

$$\pi_n = n^{(2\varsigma+2\gamma+\gamma\varsigma)/(2\varsigma+8\gamma+4\gamma\varsigma)}\log(n)^{2\gamma(3+\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)} \tag{3.8}$$

and

$$\Sigma_{\tilde{h}}(t) = \big(\sum_{j\in\mathbb{Z}} \mathbb{E}[\tilde{h}_0(t)\tilde{h}_j(t)^{\mathsf{T}}]\big)^{1/2}.$$

Based on this theorem, we are able to prove the following asymptotic statement for simultaneous confidence bands for $\theta(\cdot)$:

THEOREM 3.4 (Simultaneous confidence bands for $\theta(\cdot)$ and $\theta'(\cdot)$). *Let C be a fixed $k\times s$ matrix with rank $s \leq k$. Define $\hat{\theta}_{b_n,C}(t) := C^{\mathsf{T}}\hat{\theta}_{b_n}(t)$, $\hat{\theta}'_{b_n,C}(t) := C^{\mathsf{T}}\widehat{\theta}'_{b_n}(t)$ and $\theta_C(t) := C^{\mathsf{T}}\theta(t)$, $A_C(t) := V(t)^{-1}C$, $\Sigma_C^2(t) := A_C^{\mathsf{T}}(t)\Lambda(t)A_C(t)$.*

Let Assumption 2.1 be fulfilled with $r = 2 + \varsigma$ for some $\varsigma > 0$. Assume that, for $\alpha_{exp} = (2\gamma + \varsigma\gamma - \varsigma)/(\varsigma + 4\gamma + 2\gamma\varsigma)$, $\log(n)\big(b_n n^{\alpha_{exp}}\big)^{-1} \to 0$.

(i) If $nb_n^7 \log(n) \to 0$, then

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sqrt{nb_n}}{\sigma_{K,0}} \sup_{t\in\mathcal{T}_n} \Big|\Sigma_C^{-1}(t)\Big\{\hat{\theta}_{b_n,C}(t) - \theta_C(t) - b_n^2 \frac{\mu_{K,2}}{2}\theta_C''(t)\Big\}\Big| - B_K(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)), \tag{3.9}$$

(ii) If additionally, Assumption 2.2 is fulfilled and $nb_n^9 \log(n) \to 0$, then with $\hat{K}(x) = K(x)x$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sqrt{nb_n^3}\mu_{K,2}}{\sigma_{K,2}} \sup_{t\in\mathcal{T}_n} \Big|\Sigma_C^{-1}(t)\Big\{\hat{\theta}_{b_n,C}'(t) - \theta_C'(t) - b_n^2 \frac{\mu_{\hat{K},4}}{2\mu_{\hat{K},2}} C^\top bias(t)\Big\}\Big| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)), \tag{3.10}$$

where in both cases $\mathcal{T}_n = [b_n, 1 - b_n]$, $m^* = 1/b_n$ and

$$B_K(m^*) = \sqrt{2\log(m^*)} + \frac{\log(C_K) + (s/2 - 1/2)\log(\log(m^*)) - \log(2)}{\sqrt{2\log(m^*)}}, \tag{3.11}$$

with

$$C_K = \frac{\Big\{\int_{-1}^1 |K'(u)|^2 du/\sigma_{K,0}^2\pi\Big\}^{1/2}}{\Gamma(s/2)}.$$

Remark The results hold true with arbitrarily large $\gamma > 0$ if instead of Assumption 2.1, 2.2, Assumption 7.16, 7.17 is assumed.

Remark The conditions on b_n are fulfilled for bandwidths $b_n = n^{-\alpha}$, where $\alpha \in (0, 1)$ satisfies:

(i) $1/7 < \alpha < \alpha_{exp}$ in case (i),
(ii) $1/9 < \alpha < \alpha_{exp}$ in case (ii).

If $\gamma > 2\varsigma/(2 + \varsigma)$, then the bandwidths $b_n = cn^{-1/5}$ are covered.

Note that for practical use of the SCB in (3.9) and (3.10), one needs to estimate the bias term, choose a proper bandwidth b_n and estimate $\Sigma_C(t)$. Furthermore, the theoretical SCB only has slow logarithmic convergence, thus one requires huge n to achieve the desired coverage probability. To tackle these type of problems, we discuss practical issues in the next Section 4.

4. Implementational issues. In this section, we discuss some issues which arise by implementing the procedure from Theorem 3.4. We focus on estimation of $\hat{\theta}_{b_n}$ and optimization of the corresponding SCBs; the results for $\widehat{\theta}'_{b_n}$ can be obtained accordingly.

4.1. *Bias correction.* There are several possible ways to eliminate the bias term in (3.9). A natural way is to estimate $\theta''(t)$ by using a local quadratic estimation routine with some bandwidth $b_n' \geq b_n$. However the estimation of $\theta''(t)$ may be unstable due to the convergence condition $nb_n^5 \to \infty$ which may be hard to realize together with $nb_n^7 \log(n) \to 0$ from Theorem 3.4 in practice. Here instead we propose a bias correction via the following jack-knife method: We define

$$\tilde{\theta}_{b_n}(t) := 2\hat{\theta}_{b_n/\sqrt{2}}(t) - \hat{\theta}_{b_n}(t). \tag{4.1}$$

Since the weak Bahadur representation from Theorem 3.2(i) holds both for $\hat{\theta}_{b_n/\sqrt{2}}$ and $\hat{\theta}_{b_n}(t)$, we obtain

$$\sup_{t\in\mathcal{T}_n} \left|V(t)\cdot\{\tilde{\theta}_{b_n}(t) - \theta(t)\} - (nb_n)^{-1}\sum_{i=1}^n \tilde{K}_{b_n}(i/n-t)h_i\right| = O_{\mathbb{P}}(\tau_n^{(1)} + \beta_n b_n^2 + b_n^3 + (nb_n)^{-1}),$$

where $\tilde{K}(x) := 2\sqrt{2}K(\sqrt{2}x) - K(x)$. Note that the bias term of order b_n^2 is eliminated by construction. This shows that Theorem 3.4(i) still holds true for $\tilde{\theta}_{b_n}(\cdot)$ with kernel K replaced by the fourth-order kernel $\tilde{K}$ and with no bias term of order b_n^2.

4.2. *Estimation of the covariance matrix* $\Sigma_C(t)$. In this subsection, we discuss the estimation of $\Sigma_C^2(t)$ (namely, $V(t)$ and $\Lambda(t)$) since this term is generally unknown but arises in the SCB in Theorem 3.4. In Examples 5.1, 5.2 and 5.3 it can be seen that in many time series and independent regression models where the objective function ℓ is given by a (conditional) maximum likelihood approach, it holds that $\Lambda(t) = I(t)$ due to the fact that the $\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$, $i \in \mathbb{Z}$ are uncorrelated. In the case that the objective function ℓ coincides with the true log conditional likelihood, one has even $V(t) = I(t)$. As it can be seen in Examples 5.1 and 5.2, even in the misspecified case it may often hold that $V(t) = c_0 \cdot I(t)$ with some constant $c_0 > 0$ only dependent on properties of the i.i.d. innovations ζ_0 which can be calculated by further assumptions on ζ_0.

Therefore, it may often hold that $\Sigma_C^2(t) = C^\mathsf{T} V(t)^{-1}\Lambda(t)V(t)^{-1}C$ obeys one of the two equalities

$$\Sigma_C^2(t) = C^\mathsf{T} V(t)^{-1} I(t) V(t)^{-1} C, \qquad \text{or} \tag{4.2}$$

$$\Sigma_C^2(t) = C^\mathsf{T} I(t)^{-1} C/c_0 \quad \text{with some known constant } c_0. \tag{4.3}$$

We therefore focus on estimation of $V(t)$ and $I(t)$. We propose the (boundary-corrected) estimators

$$\text{(4.4)}\quad \hat{V}_{b_n}(t) := (nb_n\hat{\mu}_{K,0,b_n}(t))^{-1}\sum_{i=1}^{n} K_{b_n}(i/n-t)\nabla_\theta^2\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta}'_{b_n}(t)),$$

$$\text{(4.5)}\quad \hat{I}_{b_n}(t) := (nb_n\hat{\mu}_{K,0,b_n}(t))^{-1}\sum_{i=1}^{n} K_{b_n}(i/n-t)\nabla_\theta\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta}'_{b_n}(t)) \times\nabla_\theta\ell(Z_i^c,\hat{\theta}_{b_n}(t)+(i/n-t)\widehat{\theta}'_{b_n}(t))^{\mathsf{T}},$$

where $\hat{\mu}_{K,0,b_n}(t) := \int_{-t/b_n}^{(1-t)/b_n} K(x)dx$. The convergence of these estimators is given in the next Proposition. Note that the following Proposition also holds if $\widehat{\theta}'_{b_n}$ in (4.4) and (4.5) is replaced by 0.

PROPOSITION 4.1. *Let Assumption 2.1 hold with some $r>2$. Let $(\beta_n+b_n)\log(n)^2\to 0$. Then*

(i) $\sup_{t\in(0,1)}|\hat{V}_{b_n}(t)-V(t)| = O_{\mathbb{P}}((\log n)^{-1})$.
(ii) If $r>4$, then $\sup_{t\in(0,1)}|\hat{I}_{b_n}(t)-I(t)| = O_{\mathbb{P}}((\log n)^{-1})$.

This shows uniform consistency of $\hat{V}_{b_n}(\cdot)$, $\hat{I}_{b_n}(\cdot)$ if $(\beta_n+b_n)\log(n)^2\to 0$. Note that in (ii), we need more moments to discuss $\nabla_\theta\ell\cdot\nabla_\theta\ell^{\mathsf{T}}\in\mathcal{H}(2M_y,2M_x,\chi,\bar{\bar{C}})$ ($\bar{\bar{C}}>0$). In many special cases, this may be relaxed.

In either case (4.2) or (4.3), we define $\hat{\Sigma}_C(t)$ by replacing $V(t), I(t)$ by the corresponding estimators $\hat{V}_{b_n}(t)$, $\hat{I}_{b_n}(t)$.

If no relations are known between $V(t)$ and $\Lambda(t)$, one has to use a more general approach to estimate $\Lambda(t)$. We do not want to focus on this situation since the applications we have in mind (cf. Section 5) are kept by (4.2) or (4.3). Therefore, we only adopt a method from Zhou and Wu [63] to estimate $\Lambda(t)$. Define $\tilde{D}_i := \nabla_\theta\ell(Z_i^c,\hat{\theta}_{b_n}(i/n))$, $\tilde{Q}_i := \sum_{j=-m}^{m}\tilde{D}_{i+j}$ and $\tilde{\Phi}_i := \tilde{Q}_i\tilde{Q}_i^{\mathsf{T}}/(2m+1)$. Let τ_n be some bandwidth, and put $\gamma_n := \tau_n+(m+1)/n$. For $t\in\mathcal{I}_n := [\gamma_n, 1-\gamma_n]\subset(0,1)$, define

$$\tilde{\Lambda}(t) := \frac{\sum_{i=1}^n K_{\tau_n}(i/n-t)\tilde{\Phi}_i}{\sum_{i=1}^n K_{\tau_n}(i/n-t)}.$$

Note that $\tilde{\Lambda}(t)$ is always positive semi-definite. We have the following convergence result.

THEOREM 4.2. *Suppose that Assumption 2.1 holds with $r = 4$. Assume that $\omega_n = o(1)$, where $\omega_n = n^{1/4}\sqrt{m}\log(n)\{(nb_n)^{-1/2}\log(n) + b_n^2\}$. Then with $\rho = 1$,*

$$\sup_{t\in\mathcal{I}_n} |\tilde{\Lambda}(t) - \Lambda(t)| = O_{\mathbb{P}}\Big(\omega_n + \sqrt{\frac{m}{n\tau_n^2}} + m^{-1} + \tau_n^{\rho}\Big).$$

If additionally Assumption 2.2(B1),(B3) is fulfilled with $M' = 2M$ and $\nabla_\theta \ell$ is continuously differentiable with $\partial_{z_j}\nabla_\theta\ell \in \mathcal{H}(M_y - 1, M_x - 1, \chi', \hat{\chi}_j\bar{C})$ for all $j \in \mathbb{N}_0$, then one can choose $\rho = 2$.

Let us shortly discuss the choices of τ_n, b_n and m in the above setting. For two positive sequences (r_n), (s_n) we write $r_n \asymp s_n$ if r_n/s_n and s_n/r_n are bounded for all n large enough. If one chooses $m \asymp n^{q_1}$, $b_n \asymp n^{-q_2}$ and $\tau_n = n^{-q_3}$ with some $q_1, q_2, q_3 > 0$, we obtain from Theorem 4.2 that $\sup_{t\in\mathcal{I}_n} |\tilde{\Lambda}(t) - \Lambda(t)| = O_{\mathbb{P}}(n^{-\nu}) = O_{\mathbb{P}}((\log n)^{-1})$ with some $\nu > 0$ if $q_1/2 + 1/4 < \min\{2q_2, 1/2 - q_2/2\}$ and $q_1 < 1 - 2q_3$. In the special case $q_2 = 1/5$, this reduces to the condition $q_1 < \min\{3/10, 1 - 2q_3\}$.

4.3. *Bandwidth selection.* Based on the asymptotic result (3.2) in Theorem 3.1 under Assumption 2.1, the MSE global optimal bandwidth choice reads

$$\hat{b}_n = n^{-1/5} \cdot \Big(\frac{\sigma_{K,0}^2 \int_0^1 \operatorname{tr}(V(t)^{-1}I(t)V(t)^{-1})dt}{\mu_{K,2}^2 \int_0^1 |\theta''(t)|^2 dt}\Big)^{1/5}. \tag{4.6}$$

We therefore adapt a model-based cross validation method from Richter and Dahlhaus [46], which was shown to work even if the underlying parameter curve is only Hölder continuous and $\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$ is uncorrelated. Here, we reformulate this selection procedure for the local linear setting. For $j = 1, \ldots, n$, define the leave-one-out local linear likelihood

$$L_{n,b_n,-j}^c(t, \theta, \theta') := (nb_n)^{-1} \sum_{i=1, i\neq j}^{n} K_{b_n}(i/n - t)\ell(Z_i^c, \theta + (i/n - t)\theta') \tag{4.7}$$

and the corresponding leave-one-out estimator

$$(\hat{\theta}_{b_n,-j}(t), \hat{\theta}'_{b_n,-j}(t)) = \operatorname*{argmin}_{\theta\in\Theta, \theta'\in\Theta'} L_{n,b_n,-j}^c(t, \theta, \theta').$$

The bandwidth $\hat{b}_n^{CV}$ is chosen via minimizing

$$CV(b_n) := n^{-1} \sum_{i=1}^{n} \ell(Z_i^c, \hat{\theta}_{b_n,-i}(i/n))w(i/n), \tag{4.8}$$

where $w(\cdot)$ is some weight function to exclude boundary effects. A possible choice is $w(\cdot) := \mathbb{1}_{[\gamma_0, 1-\gamma_0]}$ with some fixed $\gamma_0 > 0$. Note that it is important to use the modified local linear approach due to the different bias terms (cf. Remark 3.1). In Richter and Dahlhaus [46], it was shown that the local constant version of this procedure selects asymptotically optimal bandwidths and works even if a model misspecification is present, i.e. if the function ℓ leads to estimators $\hat{\theta}_{b_n}$ which are not consistent. This motivates that a similar behavior should hold for the local constant version.

4.4. *Bootstrap method.* The SCB for $\theta_C(t)$ obtained in Theorem 3.4 provides a slow logarithmic rate of convergence to the Gumbel distribution. Thus, for even moderately large values of sample size n, it is practically infeasible to use such a theoretical SCB as the coverage will possibly be lower than the specified nominal level. We circumvent this convergence issue in this subsection by proposing a wild bootstrap algorithm. Recall the jackknife-based bias corrected estimator of $\tilde{\theta}_{b_n}$ from (4.1). Let $\tilde{\theta}_C(t) = C^T \tilde{\theta}_{b_n}(t)$. We have the following proposition as the key idea behind the bootstrap method.

PROPOSITION 4.3. *Suppose that Assumption 2.1 holds with $r = 2 + \varsigma$. Furthermore, assume that $b_n = O(n^{-\kappa})$ with $1/7 < \kappa < (2\gamma + \varsigma\gamma - \varsigma)/(\varsigma + 4\gamma + 2\gamma\varsigma)$. Then on a richer probability space, there are i.i.d. $V_1, V_2, \ldots, \sim N(0, Id_s)$ such that*

$$\sup_{t \in \mathcal{T}_n} |\hat{\theta}_{b_n,C}(t) - \theta_C(t) - \Sigma_C(t) Q_{b_n}^{(0)}(t)| = O_{\mathbb{P}}\Big(\frac{n^{-\nu}}{\sqrt{nb_n}\log(n)^{1/2}}\Big), \tag{4.9}$$

where $\nu = \min\{(2\gamma + \varsigma\gamma - \varsigma)/(2\varsigma + 8\gamma + 4\gamma\varsigma) - \kappa/2, 7\kappa/2 - 1/2, \kappa/2\} > 0$ and

$$Q_{b_n}^{(0)}(t) = \frac{1}{nb_n} \sum_{i=1}^{n} V_i K_{b_n}(i/n - t).$$

The proof of Proposition 4.3 is immediate from the approximation rates (7.68), (7.69), (7.70) and (7.72) which, ignoring the $\log(n)$ terms, are of the form $c_n \cdot (nb_n)^{-1/2} \log(n)^{-1/2}$ with

$$c_n \in \{\big(b_n n^{(2\gamma+\varsigma\gamma-\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)}\big)^{-1/2}, b_n^{1/2}, b_n, (nb_n^7)^{1/2}, (nb_n^2)^{-1/2}\}.$$

One can interpret (4.9) in the sense that $\Sigma_C(t) Q_{b_n}^{(0)}(t)$ approximates the stochastic variation in $\hat{\theta}_{b_n,C}(t) - \theta_C(t)$ uniformly over $t \in \mathcal{T}_n$ and thus it can be used as margin of errors to construct confidence bands, provided one can consistently estimate $\Sigma_C(t)$.

4.4.1. *Boundary considerations.* The results shown above only hold for $t \in \mathcal{T}_n$. For inference of some time series models like ARCH or GARCH, large bandwidths are needed to get sufficiently smooth and stable estimators even for a large number of observations. It seems hard to generalize the SCB result Theorem 3.4 to the whole interval $t \in (0,1)$. However it is possible to generalize the bootstrap procedure which may be more important in practice:

PROPOSITION 4.4. *Suppose that the conditions on κ, ν of Proposition 4.3 hold. Then on a richer probability space, there exist i.i.d. $V_1, V_2, \ldots, \sim N(0, Id_s)$ such that*

$$\sup_{t\in(0,1)} |N_{b_n}^{(0)}(t) \cdot \big\{\hat{\theta}_{b_n,C}(t) - \theta_C(t)\big\} + b_n^2 N_{b_n}^{(1)}(t)\theta_C''(t) - \Sigma_C(t)W_{b_n}(t)| = O_{\mathbb{P}}\big(\frac{n^{-\nu}}{\sqrt{nb_n}\log(n)^{1/2}}\big),$$

where

$$W_{b_n}(t) = Q_{b_n}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n}(t)}{\hat{\mu}_{K,2,b_n}(t)} \cdot Q_{b_n}^{(1)}(t) \tag{4.10}$$

and $N_{b_n}^{(j)}(t) := \frac{\hat{\mu}_{K,j,b_n}(t)\hat{\mu}_{K,j+2,b_n}(t) - \hat{\mu}_{K,j+1,b_n}(t)^2}{\hat{\mu}_{K,2,b_n}(t)}$, $\hat{\mu}_{K,j,b_n}(t) := \int_{-t/b_n}^{(1-t)/b_n} K(x)x^j dx$,

$$Q_{b_n}^{(j)}(t) = \frac{1}{nb_n}\sum_{i=1}^{n} V_i K_{b_n}(i/n - t)\big[(i/n-t)b_n^{-1}\big]^j, \qquad (j = 0, 1).$$

Note that the additional term in (4.10) reduces to $Q_{b_n}^{(0)}(t)$ for $t \in \mathcal{T}_n$.

To eliminate the bias inside $t \in \mathcal{T}_n$ it is still recommended to use the jack-knife estimator $\tilde{\theta}_C(t)$. From Proposition 4.4 we obtain

$$\begin{aligned}\sup_{t\in(0,1)} \big|N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)\big\{\tilde{\theta}_C(t) - \theta(t)\big\} + b_n^2\big\{N_{b_n/\sqrt{2}}^{(1)}(t)N_{b_n}^{(0)}(t) - N_{b_n}^{(1)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)\big\}\theta_C''(t)\\ -\Sigma_C(t)W_{b_n}^{(debias)}(t)\big| = O_{\mathbb{P}}\big(\frac{n^{-\nu}}{\sqrt{nb_n}\log(n)^{1/2}}\big),\end{aligned} \tag{4.11}$$

where

$$W_{b_n}^{(debias)}(t) = 2N_{b_n}^{(0)}(t)\cdot\big[Q_{b_n/\sqrt{2}}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n/\sqrt{2}}(t)}{\hat{\mu}_{K,2,b_n/\sqrt{2}}(t)}Q_{b_n/\sqrt{2}}^{(1)}(t)\big] - N_{b_n/\sqrt{2}}^{(0)}(t)\cdot\big[Q_{b_n}^{(0)}(t) - \frac{\hat{\mu}_{K,1,b_n}(t)}{\hat{\mu}_{K,2,b_n}(t)}Q_{b_n}^{(1)}(t)\big]. \tag{4.12}$$

The additional factor $N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)$ in (4.11) serves as an indicator how near t is to the boundary. For $t \in \mathcal{T}_n$, this factor is 1 while for $t \in (0,1)\backslash\mathcal{T}_n$, $N_{b_n}^{(0)}(t)N_{b_n/\sqrt{2}}^{(0)}(t)$ may be very small, inducing large diameters of the band near the boundary. Note that the bias correction of the jack-knife estimator $\tilde{\theta}_C(t)$ may be useless in $t \in (0,1)\backslash\mathcal{T}_n$ since $N^{(1)}_{b_n/\sqrt{2}}(t)N^{(0)}_{b_n}(t) \neq N^{(1)}_{b_n}(t)N^{(0)}_{b_n/\sqrt{2}}(t)$. However it is necessary from a theoretical point of view to use the same estimator for the whole region $(0,1)$ to get a uniform band based on the approximation (4.11).

In practice, the result (4.11) can be used as follows: We can create a large number of i.i.d. copies $W^{(boot,debias)}_{b_n}(t)$ of $W^{(debias)}_{b_n}(t)$ by creating i.i.d. copies

(4.13)
$$Q^{(0),boot}_{b_n}(t) = \frac{1}{nb_n}\sum_{i=1}^{n} V_i^* K_{b_n}(i/n-t), \qquad Q^{(1),boot}_{b_n}\frac{1}{nb_n}\sum_{i=1}^{n} V_i^* K_{b_n}(i/n-t)\cdot(i/n-t)b_n^{-1}$$

where $V_1^*, V_2^*, \dots,$ are i.i.d. $N(0, I_{s\times s})$-distributed random variables, and computing $W^{(boot,debias)}_{b_n}(t)$ according to (4.12). Quantiles of $W^{(debias)}_{b_n}(t)$ then can be determined by using the corresponding empirical quantile of the copies $W^{(boot,debias)}_{b_n}(t)$. Then one can use (4.11) to construct the confidence band for $\theta_C(t)$. For convenience of the readers, we provide a summarized algorithm of the above discussion.

Algorithm for constructing SCBs of $\theta_C(t)$:

- Compute the appropriate bandwidth b_n based on the cross validation method in Subsection 4.3 and compute $\tilde{\theta}_C(t)$ based on the jackknife-based estimator from 4.1.
- For $r = 1, \dots, N$ with some large N, generate n i.i.d. $N(0, I_{s\times s})$ random variables $V_1^*, \dots, V_n^*$ and compute $q_r = \sup_{t\in(0,1)} |W^{(boot,debias)}_{b_n}(t)|$, where $W^{(boot,debias)}_{b_n}(t)$ is computed according to (4.12), (4.13).
- Compute $u_{1-\alpha} = q_{\lfloor(1-\alpha)N\rfloor}$, the empirical $(1-\alpha)$th quantile of $\sup_{t\in[0,1]} |W^{(debias)}_{b_n}(t)|$.
- Calculate $\hat{\Sigma}_C(t) = \{C^T\hat{V}_(t)^{-1}\hat{\Lambda}(t)\hat{V}(t)^{-1}C\}^{1/2}$ with the estimators proposed in Subsection 4.2. As mentioned there, $V(t)^{-1}\Lambda(t)V(t)^{-1}$ can often be simplified.
- The SCB for $\theta_C(t)$ is $\tilde{\theta}_{C,b_n}(t) + \hat{\Sigma}_C(t)u_{1-\alpha}\mathcal{B}_s$, where $\mathcal{B}_s = \{x \in \mathbb{R}^s : |x| \leq 1\}$ is the unit ball in $\mathbb{R}^s$.

5. Examples. We now apply our theory to a large class of recursively defined time series models, GARCH processes and, as an important special case of general linear models, logistic regression models. The main goal of this chapter is to show that the theory invented in Section 3 and Section 4 covers many interesting time varying models. Due to the general formulation of the following examples, it is not possible to obtain minimal restrictions on the parameter spaces. The restrictions however can be relaxed by considering more specific models.

PROPOSITION 5.1 (Time-varying recursively defined time series models). *Assume that* $X_i = (Y_{i-1}, \ldots, Y_{i-p}, 0, \ldots)^\mathsf{T}$, $X_i^c = (Y_{i-1}, \ldots, Y_{1\vee(i-p)}, 0, \ldots)^\mathsf{T}$ *and consider*

$$Y_i = \mu(X_i, \theta(i/n)) + \sigma(X_i, \theta(i/n))\zeta_i, \tag{5.1}$$

where $\theta = (\alpha_1, \ldots, \alpha_k, \beta_0, \ldots, \beta_l)^\mathsf{T}$ *and*

$$\mu(x, \theta) := \sum_{i=1}^{k} \alpha_i m_i(x), \qquad \sigma(x, \theta) := \big(\sum_{i=0}^{l} \beta_i \nu_i(x)\big)^{1/2},$$

with some functions $m_i : \mathbb{R}^p \to \mathbb{R}$, $\nu_i : \mathbb{R}^p \to \mathbb{R}_{\geq 0}$. *Assume that*

1. ζ_i *are i.i.d. with* $\mathbb{E}\zeta_i = 0$, $\mathbb{E}\zeta_i^2 = 1$ *and for some* $a > 0$, $\mathbb{E}|\zeta_i|^{(2+a)M} < \infty$ *(*M *is defined below).*
2. *For all* $t \in [0, 1]$, *the sets*

$$\{m_1(\tilde{X}_0(t)), \ldots, m_k(\tilde{X}_0(t))\}, \qquad \{\nu_0(\tilde{X}_0(t)), \ldots, \nu_l(\tilde{X}_0(t))\}$$

are (separately) linearly independent in $\mathcal{L}_2$.

3. *There exist* $(\kappa_{ij}) \in \mathbb{R}_{\geq 0}^{k\times p}$, $(\rho_{ij}) \in \mathbb{R}_{\geq 0}^{(l+1)\times p}$ *such that for all* i*:*

$$\sup_{x\neq x'} \frac{|m_i(x) - m_i(x')|}{|x - x'|_{\kappa_{i\cdot},1}} \leq 1, \qquad \sup_{x\neq x'} \frac{|\sqrt{\nu_i(x)} - \sqrt{\nu_i(x')}|}{|x - x'|_{\rho_{i\cdot},1}} \leq 1. \tag{5.2}$$

Let $\nu_{min} > 0$ *be some constant such that for all* $x \in \mathbb{R}$, $\nu_0(x) \geq \nu_{min}$. *With some* $\beta_{min} > 0$, *choose* $\tilde{\Theta} \subset \mathbb{R}^k \times \mathbb{R}_{\geq\beta_{min}}^{l+1}$ *such that*

$$\sum_{j=1}^{p} \Big(\sup_{\theta\in\tilde{\Theta}} \sum_{i=1}^{k} |\alpha_i|\kappa_{ij} + \|\zeta_0\|_{2M} \cdot \sup_{\theta\in\tilde{\Theta}} \sum_{i=0}^{l} \sqrt{\beta_i}\rho_{ij} \Big) < 1. \tag{5.3}$$

4. *Assumption 2.1 (A2) is valid with some* $\Theta \subset \tilde{\Theta}$.

Then Assumption 2.1 is fulfilled with some $r > 2$ *for* ℓ *chosen to be proportional to the negative log Gaussian conditional likelihood,*

$$\ell(y, x, \theta) = \frac{1}{2}\Big[\Big(\frac{y - \mu(x, \theta)}{\sigma(x, \theta)}\Big)^2 + \log \sigma(x, \theta)^2\Big],$$

with $M = 3$, geometrically decaying χ and $\Lambda(t) = I(t)$. In the special case $\sigma(x,\theta)^2 \equiv \beta_0$, one can choose $M = 2$.
If (i) $\mathbb{E}\zeta_0^3 = 0$, or (ii) $\mu(x,\theta) \equiv 0$ or (iii) $\sigma(x,\theta) \equiv \beta_0$ and $\mathbb{E}m(\tilde{X}_0(t)) = 0$, then

$$I(t) = \begin{pmatrix} I_k & 0 \\ 0 & (\mathbb{E}\zeta_0^4 - 1)I_{l+1}/2 \end{pmatrix} \cdot V(t),$$

where I_d denotes the d-dimensional identity matrix.
If additionally, Assumption 2.2 (B1) is fulfilled and m_i, ν_i are differentiable such that for all $j = 1,\dots,p$ and all i,

$$\sup_{x\neq x'} \frac{|\partial_{x_j} m_i(x) - \partial_{x_j} m_i(x')|}{|x - x'|_1} < \infty, \qquad \sup_{x\neq x'} \frac{|\partial_{x_j} \nu_i(x) - \partial_{x_j} \nu_i(x')|}{|x - x'|_1} < \infty,$$

then Assumption 2.2 is fulfilled for ℓ.

In the tvAR model (cf. [46], Example 4.1), it holds that $p = k$, $m_1(x) = x_1$, ..., $m_k(x) = x_k$, $l = 0$, $\nu_0(x) = 1$, leading to the rather strong condition $\sup_{\theta\in\Theta}\sum_{i=1}^k |\alpha_i| < 1$. As seen in the proof of Proposition 5.1, the condition (5.3) however is only needed to guarantee the existence of the process and corresponding moments. By using techniques which are more specific to the model, one can obtain much less strict assumptions such as Θ being a compact subset of

$$\{\theta = (\alpha_1, ..., \alpha_k, \beta_0) \in \mathbb{R}^k \times (0,\infty) : \alpha(z) = 1 + \sum_{i=1}^k \alpha_i z^i \text{ has only zeros outside the unit circle}\},$$

cf. [46], Example 4.1.

In the tvARCH case, the above Proposition 5.1 asks for $\mathbb{E}|\zeta_i|^{6+a} < \infty$ with some $a > 0$. In the following, we consider the tvGARCH model, showing that by using matrix arguments and a more refined set of assumptions (cf. Assumption 7.16, 7.17 in the appendix), these condition can be relaxed to $\mathbb{E}|\zeta_i|^{4+a} < \infty$.

The tvGARCH model was for instance studied in the stationary case in Francq and Zakoïan [23]. More recently, pointwise asymptotic results were obtained in Rohan and Ramanathan [48]. For a matrix A, we define $\|A\|_q := (\|A_{ij}\|_q)_{ij}$ as a component-wise application of $\|\cdot\|_q$. Recall the Kronecker product from (3.1).

PROPOSITION 5.2 (tvGARCH models). *For $i = 1,\dots,n$, consider the recursion*

$$\begin{aligned} Y_i &= \sigma_i^2 \zeta_i^2, \\ \sigma_i^2 &= \alpha_0(i/n) + \sum_{j=1}^m \alpha_j(i/n) Y_{i-j} + \sum_{j=1}^l \beta_j(i/n)\sigma_{i-j}^2, \end{aligned}$$

where $\theta = (\alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_l) : [0,1] \to \Theta \subset \mathbb{R}^{m+l+1}$. *Let* $f(\theta) = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_l)^\mathsf{T}$ *and let* $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^\mathsf{T}$ *be the unit column vector with* j*th element being 1,* $1 \leq j \leq l+m$. *Define* $M_i(\theta) = (f(\theta)\zeta_i^2, e_1, \ldots, e_{m-1}, f(\theta), e_{m+1}, \ldots, e_{m+l-1})^\mathsf{T}$. *Let* $\alpha_{min} > 0$, *and*

$$\tilde{\Theta} = \{\theta \in \mathbb{R}_{\geq 0}^{m+l+1} : \alpha_0 \geq \alpha_{min}, \lambda_{max}(\|M_0(\theta)\|_2) < 1\}.$$

Suppose that

(i) Assumption 2.1(A2) is fulfilled with $\Theta \subset \mathrm{int}(\tilde{\Theta})$ *and each component of* $\theta(\cdot)$ *is in* $C^4[0,1]$,
(ii) ζ_i *are i.i.d. with* $\mathbb{E}\zeta_i = 0$, $\mathbb{E}\zeta_i^2 = 1$ *and* $\mathbb{E}|\zeta_i|^{4+a} < \infty$ *with some* $a > 0$.

Then Assumption 7.16 is fulfilled with some $r > 2$ *and the choices* $X_i = (Y_{i-1}, Y_{i-2}, \ldots)$ *and* $X_i^c = (Y_{i-1}, Y_{i-2}, \ldots, Y_1, 0, 0, \ldots)$ *for the conditional quasi likelihood*

$$\ell(y, x, \theta) = \frac{1}{2}\Big[\frac{y}{\sigma(x,\theta)^2} + \log(\sigma(x,\theta)^2)\Big],$$

where $\sigma(x,\theta)^2$ *is recursively defined via* $\sigma(x,\theta)^2 = \alpha_0 + \sum_{j=1}^m \alpha_j x_j + \sum_{j=1}^l \beta_j \sigma(x_{j\to}, \theta)^2$ *and* $x_{j\to} := (x_{j+1}, x_{j+2}, \ldots)$. *It holds that* $\Lambda(t) = I(t) = ((\mathbb{E}\zeta_0^4 - 1)/2)V(t)$.

In the important GARCH(1,1) case, the parameter space condition translated to

$$\lambda_{max}(\|M_0(\theta)\|_2) = \beta_1 + \alpha_1\|\zeta_0\|_4^2 < 1. \tag{5.4}$$

If $\zeta_0 \sim N(0,1)$, it holds that $\|\zeta_0\|_4^2 = \sqrt{3} \approx 1.73$. Bollerslev [6] proved that stationary GARCH processes have 4th moments under the condition $\lambda_{max}(\mathbb{E}[M_0(\theta)^{\otimes 2}]) < 1$ or equivalently,

$$\beta_1^2 + 2\|\zeta_0\|_2^2\alpha_1\beta_1 + \alpha_1^2\|\zeta_0\|_4^4 < 1,$$

By using $\|\zeta_0\|_2 \leq \|\zeta_0\|_4$, it is easily seen that (5.4) is slightly more restrictive. This is due to the additional approximation arguments we have to use in the case of local stationarity.

Lastly, let us consider a locally stationary logistic regression model which could be used to check if effects of certain covariates change over time. In the stationary setting, such models were for instance considered by [21], [40] and [22]. We only consider one population of size m for simplicity.

PROPOSITION 5.3 (Logistic regression). *Fix $m \in \mathbb{N}$. Assume that $\zeta_i = (\zeta_{i,0}, \zeta_{i,1}, \dots, \zeta_{i,m})^\mathsf{T}$, where $\zeta_{i,j}$, $i \in \mathbb{Z}$, $j = 0, \dots, m$ are i.i.d. uniformly distributed on $[0,1]$. Let $X_i \in \mathbb{R}^p$ be a vector of covariates, $X_i = G(i/n, \mathcal{G}_i)$ with $\mathcal{G}_i = (\dots, \zeta_{i-1,0}, \zeta_{i,0})$. For $i = 1, \dots, n$,*

$$Y_i = \sum_{j=1}^{m} \mathbb{1}_{\{\zeta_{i,j} \le \pi(X_i^\mathsf{T}\theta(i/n))\}}, \qquad i.e. \qquad Y_i|X_i \sim Bin\big(m, \pi(X_i^\mathsf{T}\theta(i/n))\big),$$

where $\pi(w)$ is given by $logit(w) = w$ and $\theta : [0,1] \to \Theta \subset \mathbb{R}^p$ is the parameter curve which we want to estimate.

We use the typical maximum likelihood approach based on

$$\ell(y, x, \theta) = m \cdot \log\big(1 + \exp\big(x^\mathsf{T}\theta\big)\big) - y \cdot \big(x^\mathsf{T}\theta\big).$$

Assume that:

1. *Assumption 2.1(A2) is fulfilled with some compact $[-D, D]^{p+1} \subset \Theta \subset \mathbb{R}^{p+1}$, $D > 0$,*
2. *For some $a > 0$, $\tilde{X}_i(t) = G(t, \mathcal{G}_i)$ fulfills $\sup_{t\in[0,1]} \|\tilde{X}_0(t)\|_{4+a} < \infty$ and $\sup_{t\in[0,1]} \delta_{4+a}^{\tilde{X}(t)}(k) = O(k^{-(1+\gamma)})$ with some $\gamma > 1$.*
3. *For all $t, t' \in [0,1]$ it holds that, with some constant $C_B > 0$,*

$$\|\tilde{X}_0(t) - \tilde{X}_0(t')\|_{4+a} \le C_B|t - t'|.$$

4. *For each $t \in [0,1]$, $\mathbb{E}[\tilde{X}_0(t)\tilde{X}_0(t)^\mathsf{T}]$ is positive definite.*

Then Assumption 2.1 is fulfilled with some $r > 2$ and $\Lambda(t) = I(t) = V(t)$.

Note that it is not possible to fulfill Assumption 2.2 in our setting of Example 5.3 since the condition of the existence of an a.s. derivative of $t \mapsto \tilde{X}_0(t)$ is too strong. It was discussed in Dahlhaus, Richter and Wu [14], that differentiability in L^1 should be enough to show the bias expansions for which Assumption 2.2 is needed, i.e. we conjecture that the results for $\widehat{\theta}'_{b_n}$ of this paper also hold true for this example.

6. Simulation results and applications. This section consists of some summarized simulations and some real data applications related to our theoretical results. Because of the generality of our theoretical framework, it is impossible to report simulation performance even for the most prominent examples in these different classes. Therefore we restrict ourselves to conditional heteroscedasticity (CH) models for simulations and real data applications. For the time-varying simultaneous band, to the best of our knowledge, there is no or little simulation results reported. For the tvAR, tvMA, tvARMA and tvRegressions we obtained satisfactory results but they are omitted here to keep this discussion concise.

6.1. *Simulations.* In this section, we study the finite sample coverage probabilities of our SCBs for theoretical coverage $\alpha = 0.9$ and $\alpha = 0.95$ in the following tvARCH(1) and tvGARCH(1,1) models:

(a) $X_i = \sqrt{\alpha_0(i/n) + \alpha_1(i/n)X_{i-1}^2}\zeta_i$, where $\alpha_0(t) = 0.8 + 0.3\cos(\pi t)$, $\alpha_1(t) = 0.45 + 0.1\cos(\pi t)$,

(b) $X_i = \sigma_i\zeta_i$, $\sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)X_{i-1}^2 + \beta_1(i/n)\sigma_{i-1}^2$, where $\alpha_0(t) = 1.3 + 0.2*\sin(2\pi t)$, $\alpha_1(t) = 0.25 + 0.05*\sin(\pi t)$ and $\beta_1(t) = 0.4 + 0.1\sin(\pi t)$,

where ζ_i is i.i.d. standard normal distributed. For estimation, we choose $K(x) = \frac{3}{4}(1 - x^2)\mathbb{1}_{[-1,1]}(x)$ to be the Epanechnikov kernel, $n = 2000$ for (a) and $n = 5000$ for (b) and b_n ranging from 0.175 to 0.375 in steps of 0.025 (the optimal bandwidths (4.6) are given by $\hat{b}_n^{(a)} \approx 0.27$ for model (a) and by $\hat{b}_n^{(b)} \approx 0.41$ for $n = 5000$ for model (b)). For each situation, $N = 2000$ replications are performed and it is checked if the obtained SCB based on (4.11) contains the true curves in $t \in (0,1)$. In both models we have $\Lambda(t) = I(t) = V(t)$ and therefore estimate $\Sigma_C^2(t) = C^{\mathsf{T}} I(t)^{-1} C$ via replacing $I(t)$ by $\hat{I}_{b_n}(t)$ from (4.5). We obtained the results given in Tables 1 and 2. The estimation, for smaller sample sizes n, sometimes may lead to difficulties since the optimization routine (*optim* in programming language R) may not converge. We decided to discard these pathological cases for simplicity. It can be seen that the empirical coverage probabilities are reasonably close to the nominal level for bandwidths close to the optimal ones and they do not differ too much for other bandwidths as well.

TABLE 1
Coverage probabilities of the SCB in (a) for $n = 1000, 2000$ *and* 5000*;* $b_n^{opt} = 0.48, 0.42, 0.34$ *respectively*

		$\alpha = 90\%$			$\alpha = 95\%$		
n	b_n	α_0	α_1	$(\alpha_0, \alpha_1)^{\mathsf{T}}$	α_0	α_1	$(\alpha_0, \alpha_1)^{\mathsf{T}}$
1000	0.4	0.873	0.846	0.845	0.937	0.906	0.900
	0.45	0.885	0.875	0.879	0.941	0.925	0.927
	0.5	0.887	0.876	0.864	0.948	0.926	0.931
	0.55	0.871	0.870	0.866	0.931	0.925	0.921
2000	0.3	0.893	0.861	0.868	0.946	0.924	0.930
	0.35	0.886	0.872	0.866	0.938	0.928	0.921
	0.4	0.891	0.878	0.874	0.937	0.926	0.933
	0.45	0.874	0.873	0.883	0.940	0.937	0.937
5000	0.25	0.885	0.883	0.882	0.941	0.931	0.936
	0.3	0.892	0.883	0.889	0.949	0.938	0.941
	0.35	0.900	0.891	0.894	0.948	0.945	0.938
	0.4	0.900	0.899	0.894	0.953	0.947	0.937
	0.45	0.878	0.880	0.881	0.934	0.937	0.930

TABLE 2
Coverage probabilities of the SCB in (b) for $n = 2000, 5000$, $b_n^{opt} = 0.49, 0.41$ *respectively*

		$\alpha = 90\%$				$\alpha = 95\%$			
n	b_n	α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^\mathsf{T}$	α_0	α_1	β_1	$(\alpha_0, \alpha_1, \beta_1)^\mathsf{T}$
2000	0.35	0.897	0.876	0.899	0.807	0.936	0.920	0.942	0.859
	0.40	0.886	0.906	0.898	0.838	0.924	0.942	0.936	0.890
	0.45	0.868	0.899	0.871	0.835	0.919	0.939	0.923	0.890
	0.50	0.881	0.902	0.890	0.831	0.926	0.945	0.935	0.885
	0.55	0.875	0.896	0.887	0.812	0.928	0.939	0.935	0.881
	0.60	0.864	0.913	0.881	0.798	0.923	0.956	0.934	0.864
5000	0.30	0.894	0.876	0.896	0.825	0.934	0.933	0.931	0.891
	0.35	0.892	0.870	0.896	0.841	0.940	0.925	0.939	0.894
	0.40	0.903	0.896	0.892	0.847	0.950	0.941	0.944	0.914
	0.45	0.884	0.902	0.887	0.850	0.933	0.946	0.938	0.906
	0.50	0.887	0.905	0.890	0.833	0.942	0.949	0.946	0.901
	0.55	0.864	0.893	0.902	0.785	0.933	0.935	0.949	0.873

6.2. *Applications.* In this section, we consider a few real-data applications of our procedure. As mentioned in Section 1, there are abundant results in the literature about time-varying regression but the results for time-varying autoregressive conditional heteroscedastic models are scarce. Thus it is important to evaluate the performance of our constructed SCBs for these type of models in both theoretical and real data scenarios. Among the popular heteroscedastic models, usually GARCH type models are most difficult to estimate due to the recursion of the variance term.

We consider two examples from the class of conditional heteroscedastic models with two types of financial datasets: one foreign exchange and one stock market daily pricing data. As Fryzlewicz, Sapatinas and Subba Rao [24] found out, ARCH models have good forecasting ability for currency exchange type data whereas for data coming from the stock market, GARCH models are preferred. Typically, these daily closing price data show unit root behavior and thus instead of using the daily price data, we model the log-return data. The log-return is defined as follows and is close to the relative return

$$Y_i = \log P_i - \log P_{i-1} = \log\left(1 + \frac{P_i - P_{i-1}}{P_{i-1}}\right) \approx \frac{P_i - P_{i-1}}{P_{i-1}},$$

where P_i is the closing price on the i^{th} day. Because of the apparent time-varying nature of volatility these log-return data typically show, conditional heteroscedastic models are used for analysis and forecasting.

6.2.1. *Real data application I: USD/GBP rates.* For the first application, we consider a tvARCH(p) model with $p = 1, 2$. It has the following form

$$Y_i^2 = \sigma_i^2 \zeta_i^2, \qquad \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)Y_{i-1}^2 + \ldots + \alpha_p(i/n)Y_{i-p}^2.$$

Many different exchange rates from 1990-1999 for USD with other currencies were analyzed in [24] using tvARCH(p) models with $p = 0, 1, 2$. The authors suggested choosing $p = 1$ for USD-GBP exchange rates. We collect the same data from `www.federalreserve.gov/releases/h10/Hist/default1999.htm` and fit both tvARCH(1) and tvARCH(2) models. This is a sample of size 2514 and we use cross-validated bandwidth 0.15 and 0.16 for the two models. We only show the results for the fit with tvARCH(1) here. We observed that the estimates for the parameter curves $\alpha_0(\cdot)$ and $\alpha_1(\cdot)$ for tvARCH(2) model are very similar to that from the tvARCH(1) fit and thus it indicates against including the extra $\alpha_2(\cdot)$ parameter in our model. We also provide the plots for the log-returns and ACF plot of squared sample that shows evidence of conditional heteroscedasticity.

Based on Figure 1 time-constancy for both the parameter curves is rejected at 5% level of significance. For $\alpha_1(\cdot)$, the estimate generally stays below the stationary fit. Also, one can see from the plot of actual log-returns that there are large shocks from 1990 to 1993 compared to those seen in 1993-1999. This can be explained through the high (low) values shown for the estimated curve $\alpha_0(\cdot)$ for the time-period 1990-1993 (1993-1999).

6.2.2. *Real data application II: Merval index data.* In the empirical analysis of log-return for stock market data, however, as Palm [43] and others suggest, lower order GARCH have been often found to account sufficiently for the conditional heteroscedasticity. Moreover, GARCH(1,1) and in a very few cases GARCH(1,2) and GARCH(2,1) models are used and higher order GARCH models are typically not necessary. Another advantage of using GARCH(1,1) over ARCH(p) models is that one need not worry about choosing a proper lag p as GARCH(1,1) can be thought as an ARCH model with $p = \infty$. In this subsection, we implement a time-varying version of GARCH(1,1) and obtain the bootstrapped SCB. A tvGARCH(1,1) model has the following form:

$$Y_i^2 = \sigma_i^2 \zeta_i^2, \qquad \sigma_i^2 = \alpha_0(i/n) + \alpha_1(i/n)Y_{i-1}^2 + \beta_1(i/n)\sigma_{i-1}^2.$$

As our second example, we choose to analyze the log returns of Merval index data from Argentina for the time period January 2010 to December 2017. In Tagliafichi Ricardo [50], the author considered daily returns for the period 1990-2000 and mentioned how

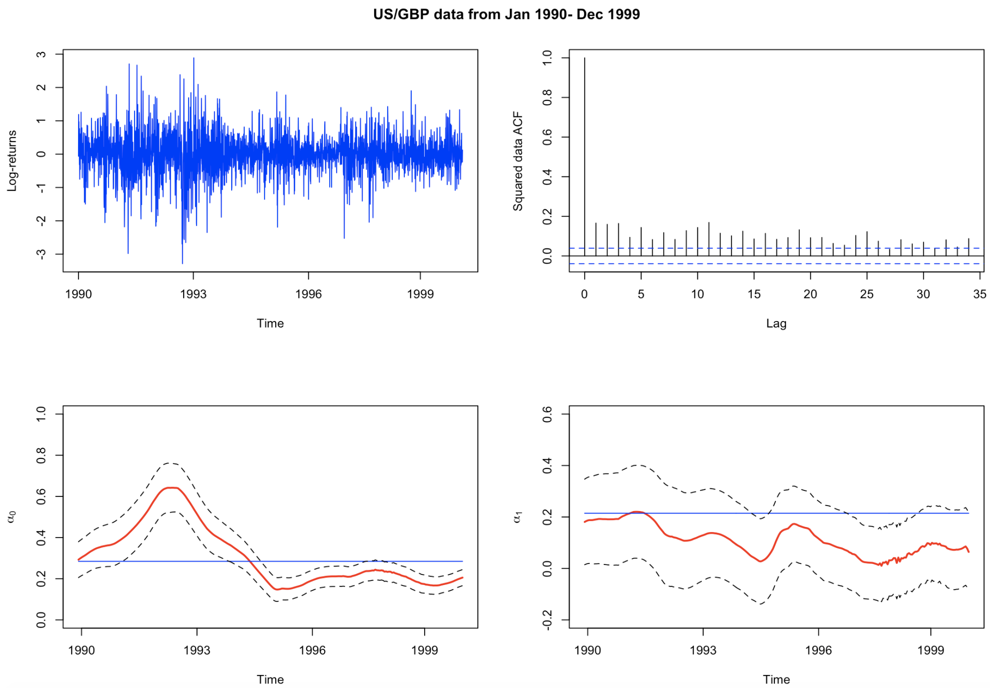

FIG 1. *Analysis of USD/GBP data from Jan 1990 to Dec 1999. Top left: Log-returns. Top right: ACF plot. Bottom panel: Estimates of the parameters $\alpha_0(\cdot)$, $\alpha_1(\cdot)$, respectively (red) with 95% SCBs (dashed) and estimates of the parameters assuming constancy (blue).*

time-varying nature can be present in the parameters of the GARCH(1,1) model he fits. In particular, he chose to split this time horizon in 3 parts and computed the estimates separately to compare with the overall estimates. This index was remodelled in 2000 and has increased about 1000% in each five years. We considered daily log returns from January 2010 to December 2017 in this analysis. Our cross-validated bandwidth is 0.445 for this data of size 1960. As one can see from Figure 2, the time series show significant lags in its ACF plot after squaring; indicating conditional heteroscedasticity.

One can see that the estimates for $\beta_1(\cdot)$ is below the corresponding time-constant fit. For the $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ parameters since it is possible to find a horizontal line passing through the corridor created by the bands, the hypothesis of time-constancy cannot be rejected at 5% level of significance. But specific patterns such as those seen in the simultaneous bands for $\alpha_1(\cdot)$ and $\beta_1(\cdot)$ cannot be implied from just a time-constant fit.

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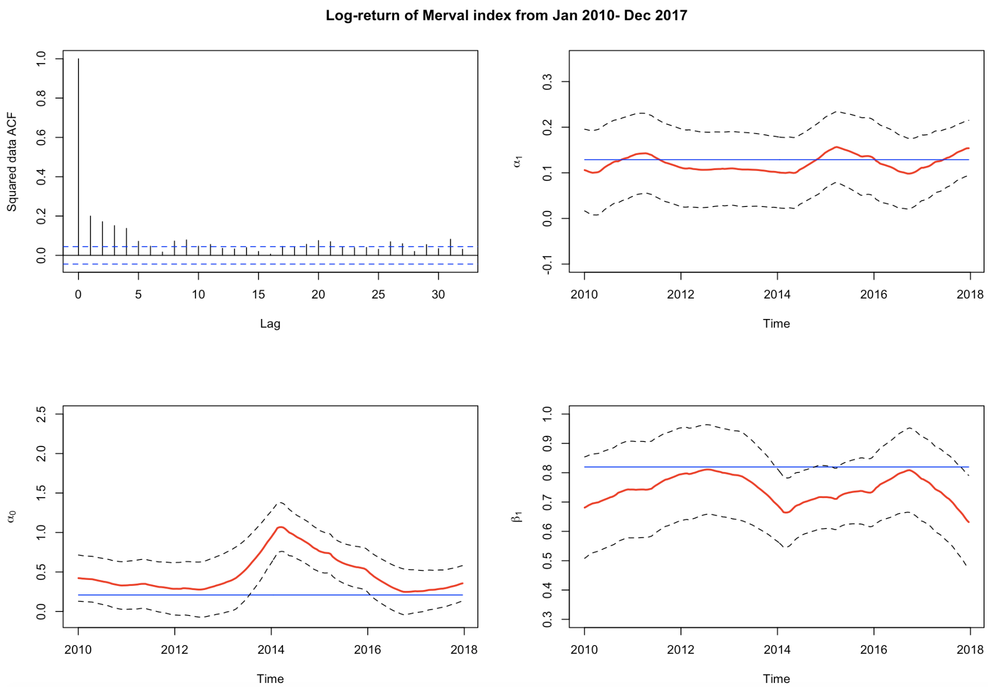

FIG 2. *Analysis of MERVAL index data from Jan 2010 to Oct 2017. Top left: ACF plot. Top right, bottom left, bottom right: Estimates of the parameters $\alpha_0(\cdot)$, $\alpha_1(\cdot)$ and $\beta_1(\cdot)$, respectively (red) with SCBs (dashed) and estimates of the parameters assuming constancy (blue).*

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Supplement: This material ([doi]COMPLETED BY THE TYPESETTER) contains the proofs of the results in the paper as well as the proofs of the examples.

7. Proofs. For $\eta = (\eta_1, \eta_2) \in \Theta \times (\Theta' \cdot b_n) =: E_n$, define

$$L_{n,b_n}^{\circ,c}(t,\eta) := (nb_n)^{-1}\sum_{i=1}^{n} K_{b_n}(i/n-t)\ell(Z_i^c, \eta_1 + \eta_2(i/n-t)b_n^{-1})$$

and $\hat{L}_{n,b_n}^{\circ}$, L_{n,b_n}° similarly as $L_{n,b_n}^{\circ,c}$ but with Z_i^c replaced by $\tilde{Z}_i(i/n)$ or Z_i, respectively. We define $\eta_{b_n}(t) = (\theta(t)^{\mathsf{T}}, b_n\theta'(t)^{\mathsf{T}})^{\mathsf{T}}$ as the value which should be estimated by $\hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^{\mathsf{T}}, b_n\widehat{\theta}_{b_n}'(t)^{\mathsf{T}})^{\mathsf{T}}$, the minimizer of $L_{n,b_n}^{\circ}(t,\eta)$. In the proof of Theorem 3.1, it is shown that $L_{n,b_n}^{\circ}(t,\eta)$ converges to $L^{\circ}(t,\eta) := \int_{-1}^{1} K(x)L(t,\eta_1+\eta_2 x)dx$. If $\chi \in \mathbb{R}^{\mathbb{N}}$, recall that $\hat{\chi} = (1,\chi) \in \mathbb{R}^{\mathbb{N}_0}$.

For $t \in (0,1)$ and $\eta \in E_n = \Theta \times (\Theta' \cdot b_n)$ and some Lipschitz continuous function $\hat{K}$ (Lipschitz constant $L_{\hat{K}}$) and compact support $[-1,1]$ ($\hat{K}$ bounded by $|\hat{K}|_\infty$), define $\hat{K}_{b_n}(\cdot) := \hat{K}(\cdot/b_n)$ and

(7.1)

$$G_n(t,\eta) := (nb_n)^{-1}\sum_{i=1}^{n} \hat{K}_{b_n}(i/n-t)\cdot\{g(Z_i,\eta_1+\eta_2(i/n-t)b_n^{-1}) - \mathbb{E}g(Z_i,\eta_1+\eta_2(i/n-t)b_n^{-1})\}.$$

Let $G_n^c(t,\eta)$, $\hat{G}_n(t,\eta)$ denote the same quantities but with Z_i replaced by Z_i^c or $\tilde{Z}_i(i/n)$, respectively.

Assumptions 2.1, 2.2 are formulated as general as possible to cover a lot of different models. However in specific situations, the conditions therein may be too strong. Later we will introduce a different set of assumptions which is specifically designed for tvGARCH models. The results can be obtained with very similar proofs. Because of that, let us introduce the more general class $\mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ for $s \geq 0$: A function $g : \mathbb{R}\times\mathbb{R}^{\mathbb{N}}\times\Theta \to \mathbb{R}$ is in $\mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ if $\sup_{\theta\in\Theta}|g(0,\theta)| \leq \bar{C}$,

$$\sup_z \sup_{\theta\neq\theta'} \frac{|g(z,\theta)-g(z,\theta')|}{|\theta-\theta'|_1 R_{M_y,M_x}(z)^{1+s}} \leq \bar{C}$$

and

$$\sup_\theta \sup_{z\neq z'} \frac{|g(z,\theta)-g(z',\theta)|}{|z-z'|_{\hat{\chi},1}\cdot\{R_{M_y-1,M_x-1}(z)^{1+s} + R_{M_y-1,M_x-1}(z')^{1+s}\}} \leq \bar{C}.$$

Obviously, $\mathcal{H} = \mathcal{H}_0$.

7.1. *Proofs of Section 3.*

PROOF OF THEOREM 3.1. The proof is similar to the proof of Theorems 5.2 and 5.4 in [14].

(i) Fix $t \in (0,1)$. By Lemma 7.8(ii) applied to ℓ, we have

$$\sup_{\eta \in E_n} |\hat{L}^{\circ}_{n,b_n}(t,\eta) - \mathbb{E}\hat{L}^{\circ}_{n,b_n}(t,\eta)| = o_{\mathbb{P}}(1).$$

Applying Lemma 7.9 to ℓ, we obtain

$$\sup_{\eta \in E_n} |\mathbb{E}\hat{L}^{\circ}_{n,b_n}(t,\eta) - L^{\circ}(t,\eta)| = O(b_n + (nb_n)^{-1}) = o(1),$$

where $L^{\circ}(t,\eta) = \int_{-1}^{1} K(x)L(t,\eta_1 + \eta_2 x)dx$. By Lemma 7.8(i), we have

$$\Big\| \sup_{\eta \in E_n} |L^{\circ,c}_{n,b_n}(t,\eta) - \hat{L}^{\circ}_{n,b_n}(t,\eta)|\Big\|_1 = O((nb_n)^{-1}),$$

and thus

$$\sup_{\eta \in E_n} |L^{\circ,c}_{n,b_n}(t,\eta) - L^{\circ}(t,\eta)| = o_{\mathbb{P}}(1).$$

By Lemma 7.1, $\eta \mapsto L^{\circ}(t,\eta)$ is Lipschitz continuous. Since $\theta(t)$ is the unique minimizer of $\theta \mapsto L(t,\theta)$, we conclude that $(\eta_1,\eta_2) = (\theta(t),0)$ is the unique minimizer of $\eta \mapsto L^{\circ}(t,\eta)$. Since $\hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^{\mathsf{T}}, b_n\widehat{\theta}'_{b_n}(t)^{\mathsf{T}})^{\mathsf{T}}$ is a minimizer of $L^{\circ,c}_{n,b_n}(t,\eta)$, standard arguments yield

$$\hat{\eta}_{b_n}(t) = (\hat{\theta}_{b_n}(t)^{\mathsf{T}}, b_n\widehat{\theta}'_{b_n}(t)^{\mathsf{T}})^{\mathsf{T}} = (\theta(t)^{\mathsf{T}},0)^{\mathsf{T}} + o_{\mathbb{P}}(1). \tag{7.2}$$

We now show that $\widehat{\theta'}_{b_n}(t) - \theta'(t) = o_{\mathbb{P}}(1)$ if $nb_n^3 \to \infty$. The following argumentation is also a preparation for the proof of (ii),(iii). By (7.2), we have that $\hat{\eta}_{b_n}(t)$ is in the interior of $\Theta \times (\Theta' \cdot b_n)$ with probability tending to 1 (since it converges to $(\theta(t)^{\mathsf{T}},0)$ in probability), thus $\nabla_\eta L^{\circ,c}_{n,b_n}(t,\hat{\eta}_{b_n}(t)) = 0$ with probability tending to 1. By a Taylor expansion we obtain

$$\hat{\eta}_{b_n}(t) - \eta_{b_n}(t) = -\big[\nabla^2_\eta L^{\circ,c}_{n,b_n}(t,\bar{\eta}(t))\big]^{-1} \cdot \nabla_\eta L^{\circ,c}_{n,b_n}(t,\eta_{b_n}(t)), \tag{7.3}$$

with some $\bar{\eta}(t) \in \Theta \times (\Theta' \cdot b_n)$ satisfying $|\bar{\eta}(t) - \eta_{b_n}(t)| \le |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)|$. Let $V(t,\theta) := \mathbb{E}\nabla^2_\theta \ell(\tilde{Z}_0(t),\theta)$. Since $g = \nabla^2_\theta \ell \in \mathcal{H}(M_y,M_x,\chi,\bar{C})$, we can use similar arguments as in (i) (but with Lemma 7.8(ii)(c) replaced by Lemma 7.8(ii)(b) in case of Assumption 7.16) to obtain

$$\sup_{|\eta - \eta_{b_n}(t)|_1 < \iota} |\nabla^2_\eta L^{\circ,c}_{n,b_n}(t,\eta) - V^{\circ}(t,\eta)| = o_{\mathbb{P}}(1), \tag{7.4}$$

where

$$V^{\circ}(t,\eta) = \int_{-1}^{1} K(x) \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} \otimes V(t, \eta_1 + \eta_2 x) dx. \tag{7.5}$$

Let

$$V^{\circ}(t) := \begin{pmatrix} 1 & 0 \\ 0 & \mu_{K,2} \end{pmatrix} \otimes V(t). \tag{7.6}$$

From (i), we have $|\bar{\eta}(t) - \eta_{b_n}(t)| \le |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = o_{\mathbb{P}}(1)$, i.e. $\bar{\eta}_1(t) = \theta(t) + o_{\mathbb{P}}(1)$ and $\bar{\eta}_2(t) = b_n\theta'(t) + o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1)$. By continuity of $\theta \mapsto V(t,\theta)$ and (7.4), we conclude that

$$\nabla_\theta^2 L_{n,b_n}^{\circ,c}(t, \bar{\eta}(t)) = V^{\circ}(t, \bar{\eta}(t)) + o_{\mathbb{P}}(1) = V^{\circ}(t) + o_{\mathbb{P}}(1). \tag{7.7}$$

By Lemma 7.8(i), we have

$$\|\nabla_\eta L_{n,b_n}^{\circ,c}(t, \eta_{b_n}(t)) - \nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t))\|_1 = O((nb_n)^{-1}). \tag{7.8}$$

With (7.3), (7.7) and (7.8) we obtain

$$\begin{aligned}
&\begin{pmatrix} \sqrt{nb_n}(\hat{\theta}_{b_n}(t) - \theta(t)) \\ \sqrt{nb_n^3}(\widehat{\theta}'_{b_n}(t) - \theta'(t)) \end{pmatrix} = \sqrt{nb_n}(\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)) \\
= \;& -V^{\circ}(t)^{-1}\sqrt{nb_n}\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) + o_{\mathbb{P}}(1) \\
= \;& -V^{\circ}(t)^{-1}\sqrt{nb_n}\big\{\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t))\big\} \\
&-V^{\circ}(t)^{-1}\begin{pmatrix} \sqrt{nb_n}\mathbb{E}\nabla_{\eta_1}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) \\ \sqrt{nb_n^3}b_n^{-1}\mathbb{E}\nabla_{\eta_2}\hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) \end{pmatrix} + o_{\mathbb{P}}(1).
\end{aligned} \tag{7.9}$$

By (7.9), it is enough to show the two convergences in probability,

$$b_n^{-1}\big\{\nabla_{\eta_2} L_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t))\big\} = o_{\mathbb{P}}(1), \tag{7.10}$$

$$b_n^{-1}\mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) = o_{\mathbb{P}}(1). \tag{7.11}$$

By (7.50) (use Lemma 7.5(i) if Assumption 2.1 holds and Lemma 7.6(i) if Assumption 7.16 holds) from the proof of Lemma 7.8(ii), applied to each component of $\nabla_\theta \ell$ with $\hat{K}(x) = K(x)x$ and $\varsigma = 1$, we obtain

$$\big\|\nabla_{\eta_2} L_{n,b_n}^{\circ}(t, \eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^{\circ}(t, \eta_{b_n}(t))\big\|_2 = O((nb_n)^{-1/2}),$$

which shows (7.10) due to $nb_n^3 \to \infty$. Using the intermediate result (7.57) in the proof of Lemma 7.10, we have

$$\mathbb{E}\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) = O(b_n^3 + n^{-1} + (nb_n)^{-1}),$$

we obtain (7.11) due to $nb_n^3 \to \infty$, which completes the proof of (i).

(ii),(iii) Our aim is to show asymptotic normality of the term in the second to last line of (7.9). Define $U_{i,n}(t) := (K_{b_n}(i/n-t), K_{b_n}(i/n-t)(i/n-t)b_n^{-1})^\mathsf{T}$. Following the proof idea of Theorem 3(ii) in [53], let $m \geq 1$ and define

$$S_{n,b_n,m}(t) := \sum_{l=0}^{m-1}(nb_n)^{-1/2}\sum_{i=1}^n U_{i,n}(t) \otimes P_{i-l}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n-t)).$$

Recall $\eta_{b_n}(t) = (\theta(t)^\mathsf{T}, b_n\theta'(t)^\mathsf{T})^\mathsf{T}$. Write shortly LIM for $\limsup_{n\to\infty}\limsup_{m\to\infty}$. Then we have for each component $j = 1, \dots, 2d_\Theta$, that

$$\begin{aligned}
& \text{LIM}\ \|S_{n,b_n,m}(t)_j - (nb_n)^{1/2}\{\nabla_{\eta_j}\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t)) - \mathbb{E}\nabla_{\eta_j}\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t))\}\|_2 \\
\leq\ & \text{LIM}\ (nb_n)^{-1/2}\sum_{l=m}^\infty \Big\|\sum_{i=1}^n (U_{i,n}(t) \otimes P_{i-l}\nabla_\theta\ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n-t)))_j\Big\|_2 \\
=\ & \text{LIM}\ (nb_n)^{-1/2}\Big(\sum_{i=1}^n \big\|(U_{i,n}(t) \otimes P_{i-l}\nabla_\theta\ell(\tilde{Z}_i(i/n), \theta(t) + \theta'(t)(i/n-t)))_j\big\|_2^2\Big)^{1/2} \\
\leq\ & |K|_\infty \text{LIM} \sum_{l=m}^\infty \sup_{t\in[0,1]}\sup_{i,\theta}\delta_2^{\nabla_{\theta_i}\ell(\tilde{Z}(t),\theta)}(l) = 0,
\end{aligned} \tag{7.12}$$

by Lemma 7.5(i) if Assumption 2.1 holds or Lemma 7.6(i) if Assumption 7.16 holds. Define $M_i(t) := (nb_n)^{-1/2}\sum_{l=0}^{m-1} U_{i,n}(t) \otimes P_i\nabla_\theta\ell(\tilde{Z}_{i+l}((i+l)/n), \theta(t) + \theta'(t)(i/n-t))$ and $\tilde{S}_{n,b_n,m}(t) := \sum_{i=1}^n M_i(t)$. It is easy to see with Lemma 7.1 or Lemma 7.3 applied to $\nabla_\theta\ell$ that with some $\iota' > 0$ small enough,

$$\sup_{|u-v|\leq\iota'}\|P_i\nabla_\theta\ell(\tilde{Z}_0(u),\theta(v))\|_2 \leq 2\sup_{|u-v|\leq\iota'}\|\nabla_\theta\ell(\tilde{Z}_0(u),\theta(v))\|_2 < \infty. \tag{7.13}$$

Since m is finite and (7.13), we conclude that for each component $j = 1, \dots, 2d_\Theta$,

$$\|S_{n,b_n,m}(t)_j - \tilde{S}_{n,b_n,m}(t)_j\|_2 = O((nb_n)^{-1/2}). \tag{7.14}$$

Let $a = (a_1^\mathsf{T}, a_2^\mathsf{T})^\mathsf{T} \in \mathbb{R}^{d_\Theta} \times \mathbb{R}^{d_\Theta}$. We want to apply a central limit theorem for martingale differences to $a^\mathsf{T}\tilde{S}_{n,b_n,m}(t)$. Put

$$\Sigma_m := \sum_{l_1,l_2=0}^{m-1}\mathbb{E}\big[P_0\nabla_\theta\ell(\tilde{Z}_{l_1}(t),\theta(t))P_0\nabla_\theta\ell(\tilde{Z}_{l_2}(t),\theta(t))^\mathsf{T}\big] = \mathrm{Cov}\Big(\sum_{l=0}^{m-1}P_0\nabla_\theta\ell(\tilde{Z}_l(t),\theta(t))\Big).$$

Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds; note that $|i/n-t| \le b_n$ implies both $\theta(t)+\theta'(t)(i/n-t)$ and $\theta(\frac{i+l_1}{n})$ to be near $\theta(t)$) applied to $\nabla_\theta \ell$ gives:

$$\begin{aligned}
&\sup_{|i/n-t|\le b_n} \|P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}((i+l_1)/n), \theta(t)+\theta'(t)(i/n-t)) - P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}(t), \theta(t))\|_2 \\
\le\ &\sup_{|i/n-t|\le b_n} \|\nabla_\theta \ell(\tilde{Z}_0((i+l_1)/n), \theta(t)+\theta'(t)(i/n-t)) - \nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))\|_2 \\
=\ &O(b_n+n^{-1})
\end{aligned}$$

and due to (7.13),

$$\sup_i \|P_i \nabla_\theta \ell(\tilde{Z}_{i+l_2}((i+l_2)/n), \theta(t)+\theta'(t)(i/n-t))\|_2 \le 2 \sup_{|u-v|\le \iota'} \|\nabla_\theta \ell(\tilde{Z}_0(u), \theta(v))\|_2 < \infty.$$

We therefore have by Hölder's and Markov's inequality that

$$\begin{aligned}
&\sum_{i=1}^n M_i(t) M_i(t)^\mathsf{T} \\
=\ &\sum_{l_1,l_2=0}^{m-1} (nb_n)^{-1} \sum_{i=1}^n K_{b_n}(i/n-t)^2 \begin{pmatrix} 1 & (i/n-t)b_n^{-1} \\ (i/n-t)b_n^{-1} & (i/n-t)^2 b_n^{-2} \end{pmatrix} \\
&\qquad \otimes \{P_i \nabla_\theta \ell(\tilde{Z}_{i+l_1}(t), \theta(t)) \cdot P_i \nabla_\theta \ell(\tilde{Z}_{i+l_2}(t), \theta(t))^\mathsf{T}\} + O_\mathbb{P}(b_n+n^{-1}) \\
=\ &\begin{pmatrix} \sigma_{K,0}^2 & 0 \\ 0 & \sigma_{K,2}^2 \end{pmatrix} \otimes \Sigma_m + o_\mathbb{P}(1).
\end{aligned}$$

The last step is due to Lemma A.2 in [15]. It remains to show a Lindeberg-type condition for $M_i(t)$. Put $\tilde{M}_{ij,l} := P_i \nabla_{\theta_j} \ell(\tilde{Z}_{i+l}((i+l)/n), \theta(t)+\theta'(t)(i/n-t))$. There exists some constant $C>0$ such that for $j=1,\dots,d_\Theta$ and $\iota>0$,

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}\big[M_i(t)_j^2 \mathbb{1}_{\{|M_i(t)_j|>\iota\}}\big] \\
\le\ &C(nb_n)^{-1} \sum_{l=0}^{m-1} \sum_{i=1}^n K_{b_n}(i/n-t)^2 \mathbb{E}\big[\tilde{M}_{ij,l}^2 \mathbb{1}_{\{|K|_\infty |\tilde{M}_{ij,l}|>\iota(nb_n)^{1/2}\}}\big]. \qquad (7.15)
\end{aligned}$$

Using Hölder's inequality we have

$$\mathbb{E}\big[\tilde{M}_{ij,l}^2 \mathbb{1}_{\{|K|_\infty |\tilde{M}_{ij,l}|>\iota(nb_n)^{1/2}\}}\big] \le \mathbb{E}[|\tilde{M}_{ij,l}|^{2+a}]^{2/(2+a)} \mathbb{P}(|K|_\infty |\tilde{M}_{ij,l}| > \iota(nb_n)^{1/2})^{a/(2+a)},$$

which tends to zero using Markov's inequality, (7.13) (with $\|\cdot\|_2$ replaced by $\|\cdot\|_{2+a}$) and the condition

$$\sup_{j\in\mathbb{N}_0}\sup_{t\in[0,1]}\|\tilde{Z}_0(t)_j\|_{(2+a)M}<\infty$$

(which is automatically fulfilled with $a=\varsigma$ if Assumption 7.16 holds with $r=2+\varsigma$). This shows that (7.15) is tending to 0. The proof for $j=d_\Theta+1,\ldots,2d_\Theta$ is similar. From Theorem 18.2 in Billingsley [5] and the Cramer-Wold device we obtain

$$\tilde{S}_{n,b_n,m}(t)\xrightarrow{d}N\Big(0,\begin{pmatrix}\sigma_{K,0}^2 & 0\\ 0 & \sigma_{K,2}^2\end{pmatrix}\otimes\Sigma_m\Big). \tag{7.16}$$

Using Theorem 5.46 in [52], (7.12), (7.14), (7.16) and $\Sigma_m\to\Lambda(t)$ $(m\to\infty)$, we obtain

$$(nb_n)^{1/2}\{\nabla_\eta\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t))-\mathbb{E}\nabla_\eta\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t))\}\xrightarrow{d}N\Big(0,\begin{pmatrix}\sigma_{K,0}^2 & 0\\ 0 & \sigma_{K,2}^2\end{pmatrix}\otimes\Lambda(t)\Big). \tag{7.17}$$

Using (7.17), the expansion (7.9) and Lemma 7.10, we obtain the result provided that $nb_n^7\to 0$ for (ii) and $nb_n^9\to 0$ for (iii). □

PROOF OF THEOREM 3.2. (i),(ii) By Lemma 7.8(i),(iii)(a) and Lemma 7.9(a) (in case Assumption 2.1 holds) or Lemma 7.8(i),(iii)(c) and Lemma 7.9(a) (if Assumption 7.16 holds) applied to $g=\ell$, we have that

$$\sup_{t\in\mathcal{T}_n}\sup_{\eta\in E_n}|L^{\circ,c}_{n,b_n}(t,\eta)-L^\circ(t,\eta)|=O_{\mathbb{P}}(\beta_n+(nb_n)^{-1})+O(b_n),$$

i.e. $L^{\circ,c}_{n,b_n}(t,\eta)$ converges to $L^\circ(t,\eta)$ uniformly in t,η if $b_n=o(1)$ and $\beta_n=o(1)$. It was already seen in the proof of Theorem 3.1 that $L^\circ(t,\eta)$ is continuous w.r.t. η and uniquely minimized by $\eta=(\theta(t)^\intercal,0)^\intercal$. Standard arguments give

$$\sup_{t\in\mathcal{T}_n}|\hat{\eta}_{b_n}(t)-\eta_{b_n}(t)|=o_{\mathbb{P}}(1). \tag{7.18}$$

Thus for n large enough, $\hat{\eta}_{b_n}(t)$ is in the interior of E_n uniformly in t. By a Taylor expansion, we obtain for each $t\in\mathcal{T}_n$:

$$\hat{\eta}_{b_n}(t)-\eta_{b_n}(t)=-\big[V^\circ(t)+R_{n,b_n}(t)\big]^{-1}\cdot\nabla_\eta L^{\circ,c}_{n,b_n}(t,\eta_{b_n}(t)), \tag{7.19}$$

where $R_{n,b_n}(t)=\nabla^2_\eta L^{\circ,c}_{n,b_n}(t,\bar{\eta}(t))-V^\circ(t)$ with some $\bar{\eta}(t)\in E_n$ satisfying $|\bar{\eta}(t)-\eta_{b_n}(t)|_1\le|\hat{\eta}_{b_n}(t)-\eta_{b_n}(t)|_1$ and $V^\circ(t)$ is defined in (7.6).

By Lemma 7.8(i),(iii)(a) and Lemma 7.9(a) (if Assumption 2.1 holds) or Lemma 7.8(i),(iii)(c) and Lemma 7.9(b) (if Assumption 7.16 holds) applied to $g = \nabla_\theta^2 \ell$ and $\hat{K}(x) = K(x)$, $\hat{K}(x) = K(x)x$ or $\hat{K}(x) = K(x)x^2$, respectively, we have for some fixed $\iota' > 0$:

$$\sup_{t\in\mathcal{T}_n} \sup_{|\eta-\eta_{b_n}(t)|<\iota'} |\nabla_\eta^2 L_{n,b_n}^{\circ,c}(t,\eta) - V^\circ(t,\eta)| = O_{\mathbb{P}}(\beta_n + (nb_n)^{-1}) + O(b_n), \tag{7.20}$$

where $V^\circ(t,\eta)$ is defined in (7.5).

For the moment, let $\tilde{h}_i(t) = \nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$. Note that $\mathbb{E}\tilde{h}_0(t) = \mathbb{E}\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) = 0$ by Assumption 2.1(A3), (A1) (or Assumption 7.16(A3'), (A1')).

By Lemma 7.5(i) (if Assumption 2.1 holds) or Lemma 7.6(i) (if Assumption 7.16 holds), we have $\sup_t \delta_{2+\varsigma}^{\tilde{h}(t)_j}(k) = \sup_t \delta_{2+\varsigma}^{\nabla_{\theta_j}\ell(\tilde{Z}(t),\theta(t))}(k) = O(k^{-(1+\gamma)})$ for each $j = 1,\dots,d_\Theta$. Using Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds), we see that the assumptions of Lemma 7.14 are fulfilled and thus, applied to $\tilde{h}_i(t)$,

$$\sup_{t\in\mathcal{T}_n} \big|(nb_n)^{-1}\sum_{i=1}^n K_{b_n}(i/n-t)\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))\big| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n)). \tag{7.21}$$

With Lemma 7.12, we obtain

$$\sup_{t\in\mathcal{T}_n} \big|\nabla_\eta \hat{L}_{n,b_n}^\circ(t,\eta_{b_n}(t)) - \mathbb{E}\nabla_\eta \hat{L}_{n,b_n}^\circ(t,\eta_{b_n}(t))\big| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + \beta_n b_n^2).$$

Since $\mathbb{E}\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) = 0$, we obtain with Lemma 7.10(i),(ii) and Lemma 7.8(i):

$$\sup_{t\in\mathcal{T}_n} |\nabla_{\eta_j} L_{n,b_n}^{\circ,c}(t,\eta_{b_n}(t))| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^{1+j}), \tag{7.22}$$

where $j = 1,2$. Since $\theta \mapsto V(t,\theta) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t),\theta)$ is Lipschitz continuous (apply Lemma 7.1 in case of Assumption 2.1 or Lemma 7.3 in case of Assumption 7.16 to $\nabla^2\ell$), the same holds for $\eta \mapsto V^\circ(t,\eta)$. We conclude that with some constant $C > 0$,

$$\sup_{t\in\mathcal{T}_n} |R_{n,b_n}(t)| \le \sup_{t\in\mathcal{T}_n} \sup_{\eta\in E_n} |\nabla_\eta^2 L_{n,b_n}^{\circ,c}(t,\eta) - V^\circ(t,\eta)| + C \sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)|. \tag{7.23}$$

Inserting (7.22), (7.23) and (7.18) into (7.19), we obtain

$$\sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n,j}(t) - \eta_{b_n,j}(t)| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^{1+j}), \tag{7.24}$$

where $j = 1, 2$. Inserting (7.24), (7.20) into (7.23), we get $\sup_{t\in\mathcal{T}_n} |R_{n,b_n}(t)| = O_{\mathbb{P}}(\beta_n + b_n + (nb_n)^{-1})$. Together with

$$\begin{aligned}
& \big|V^{\circ}(t)\big(\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)\big) - \nabla L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t))\big| \\
\leq\ & \big|\big[I_{2k\times 2k} + V^{\circ}(t)^{-1}R_{n,b_n}(t)\big]^{-1} - I^{-1}_{2k\times 2k}\big| \cdot |\nabla_\eta L^{\circ,c}_{n,b_n}(t, \eta_n(t))| \\
\leq\ & \big|\big[I_{2k\times 2k} + V^{\circ}(t)^{-1}R_{n,b_n}(t)\big]^{-1}\big| \cdot \big|V^{\circ}(t)^{-1}R_{n,b_n}(t)\big| \cdot |\nabla_\eta L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t))|,
\end{aligned}$$

and (7.22) we have (3.4) and (3.6). The other results (3.5) and (3.7) follow from Lemma 7.8(i), Lemma 7.12 and Lemma 7.10. □

LEMMA 7.1. *Let $q > 0$ and $s \geq 0$. Let $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ and $M := \max\{M_x, M_y\}$. Let $\hat{Y}, \hat{Y}'$ be random variables and $\hat{X} = (\hat{X}_j)_{j\in\mathbb{N}}$, $\hat{X}' = (\hat{X}'_j)_{j\in\mathbb{N}}$ be sequences of random variables. Assume that there exists some $D > 0$ such that uniformly in $j \in \mathbb{N}$,*

$$\|\hat{Y}\|_{qM(1+s)}, \qquad \|\hat{Y}'\|_{qM(1+s)}, \qquad \|\hat{X}_j\|_{qM(1+s)}, \qquad \|\hat{X}'_j\|_{qM(1+s)} \leq D. \tag{7.25}$$

Let $\hat{Z} = (\hat{Y}, \hat{X})$, $\hat{Z}' = (\hat{Y}', \hat{X}')$. Then there exists some constant $C > 0$ only dependent on M, D, χ and $\tilde{D}$ (only in (ii)) such that

(i)

$$\|\sup_{\theta\in\Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q \leq \bar{C} \cdot C \sum_{j=0}^{\infty} \chi_j \|\hat{Z}_j - \hat{Z}'_j\|_{qM}, \tag{7.26}$$

$$\Big\| \sup_{\theta\neq\theta'} \frac{|g(\hat{Z}, \theta) - g(\hat{Z}, \theta')|}{|\theta - \theta'|_1} \Big\|_q \leq \bar{C} \cdot C, \tag{7.27}$$

$$\|\sup_{\theta\in\Theta} |g(\hat{Z}, \theta)|\|_q \leq \bar{C} \cdot C, \tag{7.28}$$

$$\|R_{M_x,M_y}(\hat{Z})^{1+s}\|_q \leq C. \tag{7.29}$$

(ii) Let $s = 0$. If additionally, $\mathbb{E}[|\hat{Y} - \hat{Y}'|^{qM_y}|\sigma(\hat{X}, \hat{X}')] \leq \tilde{D}|\hat{X} - \hat{X}'|^{qM_y}_{\chi,1}$ with some constant $\tilde{D} > 0$, then

$$\|\sup_{\theta\in\Theta} |g(\hat{Z}, \theta) - g(\hat{Z}', \theta)|\|_q \leq \bar{C} \cdot C \sum_{j=1}^{\infty} \chi_j \|\hat{X}_j - \hat{X}'_j\|_{qM}. \tag{7.30}$$

PROOF OF LEMMA 7.1. During the proofs, we consider $M_y, M_x \geq 2$ and thus $M \geq 2$. In the case $M_y = 1$ or $M_x = 1$, the proofs are easier since some terms do not show up.

(i) Note that R_{M_y-1,M_x-1} is a polynomial in $|x|_{\chi,1}$, $|y|$ with (joint) degree at most $M-1$. Since

$$R_{M_y-1,M_x-1}(\hat{Z}) = \sum_{k+l\le M-1, 0\le k\le M_y-1, 0\le l\le M_x-1} |\hat{Y}|^k |\hat{X}|_{\chi,1}^l,$$

we have by Hölder's inequality,

$$\begin{aligned} &\|R_{M_y-1,M_x-1}(\hat{Z})\|_{q(1+s)M/(M-1)} \\ \le\ & \sum_{k+l\le M-1, 0\le k\le M_y-1, 0\le l\le M_x-1} \big(\sum_{i=1}^{\infty}\chi_i\|\hat{X}_i\|_{q(1+s)M}\big)^l \|\hat{Y}\|_{q(1+s)M}^k \\ \le\ & \sum_{0\le k+l\le M-1} (|\chi|_1 D)^l D^k \le (1+D(|\chi|_1+1))^{M-1}. \end{aligned} \tag{7.31}$$

Therefore:

$$\begin{aligned} &\|\sup_{\theta\in\Theta}|g(\hat{Z},\theta)-g(\hat{Z}',\theta)|\|_q \\ \le\ & \bar{C}\big\||\hat{Y}-\hat{Y}'|(R_{M_y-1,M_x-1}(\hat{Y},\hat{X})^{1+s}+R_{M_y-1,M_x-1}(\hat{Y}',\hat{X})^{1+s})\big\|_q \\ & \qquad +\big\||\hat{X}-\hat{X}'|_{\chi,1}\cdot(R_{M_y-1,M_x-1}(\hat{Y},\hat{X})^{1+s}+R_{M_y-1,M_x-1}(\hat{Y},\hat{X}')^{1+s})\big\|_q \qquad (7.32) \\ \le\ & \bar{C}\|\hat{Y}-\hat{Y}'\|_{qM}\cdot\big(\|R_{M_y-1,M_x-1}(\hat{Y},\hat{X})\|_{q(1+s)M/(M-1)}^{1+s} \\ & \qquad\qquad +\|R_{M_y-1,M_x-1}(\hat{Y}',\hat{X})\|_{q(1+s)M/(M-1)}^{1+s}\big) \\ & \quad +\bar{C}\||\hat{X}-\hat{X}'|_{\chi,1}\|_{qM}\big(\|R_{M_y-1,M_x-1}(\hat{Y},\hat{X})\|_{q(1+s)M/(M-1)}^{1+s} \\ & \qquad\qquad +\|R_{M_y-1,M_x-1}(\hat{Y},\hat{X}')\|_{q(1+s)M/(M-1)}^{1+s}\big) \\ \le\ & 2\bar{C}(1+D(|\chi|_1+1))^{(M-1)(1+s)}\big(\|\hat{Y}-\hat{Y}'\|_{qM}+\sum_{j=1}^{\infty}\chi_j\|\hat{X}_j-\hat{X}_j'\|_{qM}\big), \end{aligned}$$

which shows (7.26). The proof of (7.28) is obvious from (7.26) and $\sup_{\theta\in\Theta}|g(0,\theta)|\le\bar{C}$.

R_{M_y,M_x} is a polynomial in $|x|_{\chi,1}$ and $|y|$ with (joint) degree at most M. As in (7.31), we obtain

$$\|R_{M_y,M_x}(\hat{Z})\|_{q(1+s)} \le (1+D(|\chi|_1+1))^M,$$

showing (7.29).

(7.27) follows from (7.29) and

$$|g(\hat{Z},\theta)-g(\hat{Z},\theta')| \le \bar{C}|\theta-\theta'|_1 R_{M_y,M_x}(\hat{Z})^{1+s}.$$

(ii) We first obtain (7.32) as before. The second summand has the upper bound

$$2\bar{C}(1+D(|\chi|_1+1))^{M-1}\sum_{j=1}^{\infty}\chi_j\|\hat{X}_j-\hat{X}_j'\|_{qM}.$$

For the first summand in (7.32), notice that

$$\begin{aligned}&\big\||\hat{Y}-\hat{Y}'|\cdot R_{M_y-1,M_x-1}(\hat{Y},\hat{X})\big\|_q\\ \leq\ & \sum_{k+l\leq M-1, 0\leq k\leq M_y-1, 0\leq l\leq M_x-1}\||\hat{Y}-\hat{Y}'|\cdot|\hat{Y}|^k\cdot|\hat{X}|_{\chi,1}^l\|_q.\end{aligned}$$

By Hölder's inequality for conditional expectations,

$$\begin{aligned}&\big\||\hat{Y}-\hat{Y}'|\cdot|\hat{Y}|^k|\hat{X}|_{\chi,1}^l\big\|_q\\ =\ & \mathbb{E}\big[\mathbb{E}[|\hat{Y}-\hat{Y}'|^q|\hat{Y}|^{qk}|\sigma(\hat{X},\hat{X}')]\cdot|\hat{X}|_{\chi,1}^{ql}\big]^{1/q}\\ \leq\ & \mathbb{E}[\mathbb{E}[|\hat{Y}-\hat{Y}'|^{qM_y}|\sigma(\hat{X},\hat{X}')]^{1/M_y}\mathbb{E}[|\hat{Y}|^{qkM_y/(M_y-1)}|\sigma(\hat{X},\hat{X}')]^{(M_y-1)/M_y}|\hat{X}|_{\chi,1}^{ql}]^{1/q}\\ =:\ & \mathbb{E}[A_1\cdot A_2\cdot A_3]^{1/q}.\end{aligned}$$

By the additional condition, we have $A_1\leq\tilde{D}^{1/M_y}|\hat{X}-\hat{X}'|_{\chi,1}^q$. By Hölder's inequality,

$$\mathbb{E}[A_1\cdot A_2\cdot A_3]^{1/q}\leq\mathbb{E}[A_1^M]^{1/(qM)}\mathbb{E}[A_2^{M/k}]^{k/(qM)}\mathbb{E}[A_3^{M/(M-k-1)}]^{(M-k-1)/(qM)}.$$

We have $\mathbb{E}[A_1^M]^{1/M}\leq\tilde{D}^{1/M_y}\||\hat{X}-\hat{X}'|_{\chi,1}\|_{qM}^q$,

$$\mathbb{E}[A_3^{M/(M-k-1)}]^{(M-k-1)/M}=\||\hat{X}|_{\chi,1}\|_{Mql/(M-k-1)}^{ql}\leq\||\hat{X}|_{\chi,1}\|_{qM}^{ql}$$

and by Jensen's inequality for conditional expectations (note that $\frac{M_y-1}{k}\frac{M}{M_y}\geq 1$),

$$\mathbb{E}[A_2^{M/k}]^{k/M}\leq\mathbb{E}[\mathbb{E}[|\hat{Y}|^{qkM_y/(M_y-1)\cdot\frac{M_y-1}{M_y}\frac{M}{k}}|\sigma(\hat{X},\hat{X}')]]^{k/M}=\|\hat{Y}\|_{Mq}^{qk}.$$

Putting the results together we obtain

$$\||\hat{Y}-\hat{Y}'|\cdot|\hat{Y}|^k|\hat{X}|_{\chi,1}^l\|_q\leq\tilde{D}^{1/(qM_y)}\sum_{i=1}^{\infty}\chi_i\|\hat{X}_i-\hat{X}_i'\|_{qM}\cdot D^k(|\chi|_1D)^l,$$

which leads to

$$\big\||\hat{Y}-\hat{Y}'|\cdot R_{M_y-1,M_x-1}(\hat{Y},\hat{X})\big\|_q\leq\tilde{D}^{1/(qM_y)}(1+D(1+|\chi|_1))^{M-1}\cdot\sum_{i=1}^{\infty}\chi_i\|\hat{X}_i-\hat{X}_i'\|_{qM},$$

giving the result. □

LEMMA 7.2 (for tvGARCH). *Let $q > 0$ and $s > 0$. Let $\hat{X}, \hat{X}', \hat{Y}, \hat{Y}'$ be as in Lemma 7.1 satisfying (7.25). Let $g = \ell$ satisfy (7.94). Then there exists some constant $C^{(s)} > 0$ only dependent on M, D, $\chi^{(s)}$ such that*

$$\|\sup_{\theta\in\Theta}|g(\hat{Z},\theta) - g(\hat{Z}',\theta)|\|_q \leq \bar{C}^{(s)}\cdot C^{(s)}\sum_{j=0}^{\infty}\hat{\chi}_j^{(s)}\big(\|\hat{Z}_j - \hat{Z}_j'\|_{qM} + \|\hat{Z}_j - \hat{Z}_j'\|_{qM(1+s)}^s\big), \tag{7.33}$$

$$\|\sup_{\theta\in\Theta}|g(\hat{Z},\theta)|\|_q \leq \bar{C}\cdot C. \tag{7.34}$$

PROOF OF LEMMA 7.2. It holds that

$$\begin{aligned}
&\big\|\sup_{\theta\in\Theta}|g(\hat{Z},\theta) - g(\hat{Z}',\theta)|\big\|_q\\
\leq\ & \bar{C}^{(s)}\big\||\hat{Z} - \hat{Z}'|_{\chi^{(s)},s}\cdot(R_{M,M}(\hat{Z}) + R_{M,M}(\hat{Z}'))\big\|_q\\
&\qquad + \bar{C}^{(s)}\big\||\hat{Z} - \hat{Z}'|_{\chi^{(s)},1}\cdot(R_{M-1,M-1}(\hat{Z})^{1+s} + R_{M-1,M-1}(\hat{Z}')^{1+s})\big\|_q.
\end{aligned}$$

The second summand can be dealt with as in the proof of Lemma 7.1(i), giving the upper bound

$$2\bar{C}^{(s)}(1 + D(|\chi^{(s)}|_1 + 1))^{(M-1)(1+s)}\big(\|\hat{Y} - \hat{Y}'\|_{qM} + \sum_{j=1}^{\infty}\chi_j^{(s)}\|\hat{X}_j - \hat{X}_j'\|_{qM}\big).$$

For the first summand, we obtain with Hölder's inequality:

$$\begin{aligned}
&\big\||\hat{Z} - \hat{Z}'|_{\hat{\chi}^{(s)},s}^s\cdot(R_{M,M}(\hat{Z}) + R_{M,M}(\hat{Z}'))\big\|_q\\
\leq\ & \sum_{j=0}^{\infty}\hat{\chi}_j^{(s)}\|\hat{Z}_j - \hat{Z}_j'\|_{q(M+s)}^s\cdot\big(\|R_{M,M}(\hat{Z})\|_{q(M+s)/M} + \|R_{M,M}(\hat{Z}')\|_{q(M+s)/M}\big),
\end{aligned}$$

giving the result since $M + s \leq M(1+s)$ and thus $\|R_{M,M}(\hat{Z})\|_{q(M+s)/M} \leq (1 + D(|\chi^{(s)}|_1 + 1))^M$. $\square$

LEMMA 7.3 (for tvGARCH). *Let $q > 0, s > 0, \iota > 0$. Let $\hat{X}, \hat{X}', \hat{Y}, \hat{Y}'$ be as in Lemma 7.1 satisfying (7.25). Let $g \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi, \bar{C})$. Let ζ_0 be independent of $\hat{X}$, $\hat{X}'$ with $\|\zeta_0\|_{qM} \leq D$. Then there exists some constant $C > 0$ only dependent on M, D, χ, $\bar{C}$ such* *that*

$$\|\sup_{|\theta-\tilde\theta|<\iota}|\tilde g_{\tilde\theta}(\zeta_0,\hat X,\theta)-\tilde g_{\tilde\theta}(\zeta_0,\hat X',\theta)|\|_q \leq \bar C\cdot C\sum_{j=1}^{\infty}\chi_j\|\hat X_j-\hat X_j'\|_{qM}, \tag{7.35}$$

$$\|\sup_{\theta\neq\theta',|\theta-\tilde\theta|_1<\iota,|\theta'-\tilde\theta|_1<\iota}\frac{|\tilde g_{\tilde\theta}(\zeta_0,\hat X,\theta)-\tilde g_{\tilde\theta}(\zeta_0,\hat X,\theta')|}{|\theta-\theta'|_1}\|_q \leq \bar C\cdot C, \tag{7.36}$$

$$\|\sup_{|\theta-\tilde\theta|_1<\iota}|\tilde g(\zeta_0,\hat X,\theta)|\|_q \leq \bar C\cdot C. \tag{7.37}$$

Proof of Lemma 7.3. As in the proof of Lemma 7.1(i), we have:

$$\|R_{M_y-1,M_x-1}(1,\hat X)\|_{q(1+s)M/(M-1)}\leq(1+D(|\chi|_1+1))^{M-1}.$$

Thus, with Hölder's inequality,

$$\begin{aligned}&\Big\|\sup_{|\theta-\tilde\theta|<\iota}|g(\zeta_0,\hat X,\theta)-g(\zeta_0,\hat X',\theta)|\Big\|_q\\ \leq\;& 2\bar C(1+D(|\chi|_1+1))^{(M-1)(1+s)}(1+\|\zeta_0\|_{qM})\sum_{j=1}^{\infty}\chi_j\|\hat X_j-\hat X_j'\|_{qM}.\end{aligned}$$

This shows (7.35). (7.37) follows from (7.35) since

$$\|\sup_{|\theta-\tilde\theta|<\iota}|\tilde g_{\tilde\theta}(\zeta_0,0,\theta)|\|_q\leq\bar C(1+\|\zeta_0\|_{qM}^M)\leq\bar C(1+D^M).$$

□

Lemma 7.4. *Assume that $\theta(\cdot)\in C^2[0,1]$. Let $\chi'=(\tilde\chi_i')_{i\in\mathbb N}$ be an absolutely summable sequence. Let $g:\mathbb R\times\mathbb R^{\mathbb N}\times\Theta\to\mathbb R$ be continuously differentiable. Suppose that either*

(a) Assumption 2.1(A5) holds with some $r\geq 2$ and Assumption 2.2(B3) holds. Additionally, $g\in\mathcal H(M_y',M_x',\chi,\bar C)$, $\nabla_\theta g\in\mathcal H(M_y',M_x',\chi,\bar C)$ and for all $l\in\mathbb N$, $\partial_{x_l}g\in\mathcal H(M_y'-1,M_x'-1,\tilde\chi,\bar C\chi_l)$.

or

(b) (for tvGARCH) There exists $s\geq 0$ such that Assumption 7.16(A5') holds with $r\geq 2(1+s)$ and Assumption 7.17(B3') holds. Additionally, $g\in\mathcal H_{s,\iota}^{mult}(M_y',M_x',\chi,\bar C)$, $\nabla_\theta g\in\mathcal H_{s,\iota}^{mult}(M_y',M_x',\chi,\bar C)$ and for all $l\in\mathbb N$, $\partial_{x_l}g\in\mathcal H_{s,\iota}^{mult}(M_y'-1,M_x'-1,\tilde\chi,\bar C\chi_l)$.

Define $M':=\max\{M_y',M_x'\}$. Then

(i) $\sup_{t\in[0,1]}\|\partial_t g(\tilde{Z}_0(t),\theta(t))\|_1 < \infty$,

(ii)

$$\sup_{t\neq t'}\frac{\|\partial_t g(\tilde{Z}_0(t),\theta(t)) - \partial_t g(\tilde{Z}_0(t'),\theta(t'))\|_1}{|t-t'|} < \infty.$$

Proof of Lemma 7.4. (i) Note that

$$\partial_t g(\tilde{Z}_0(t),\theta(t)) = \partial_z g(\tilde{Z}_0(t),\theta(t))\partial_t \tilde{Z}_0(t) + \nabla_\theta g(\tilde{Z}_0(t),\theta(t))\theta'(t). \tag{7.38}$$

By Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds),

$$\sup_t \|\nabla_\theta g(\tilde{Z}_0(t),\theta(t))\|_1 < \infty.$$

and there exists a constant $C > 0$ such that for each $j \in \mathbb{N}_0$,

$$\|\partial_{z_j} g(\tilde{Z}_0(t),\theta(t))\|_{M'/(M'-1)} \le C\bar{C}\hat{\chi}_j. \tag{7.39}$$

It follows that

$$\begin{aligned}\|\partial_z g(\tilde{Z}_0(t),\theta(t))\partial_t \tilde{Z}_0(t)\|_1 &\le \sum_{j=0}^{\infty}\|\partial_{z_j} g(\tilde{Z}_0(t),\theta(t))\|_{M'/(M'-1)}\cdot\|\partial_t \tilde{Z}_{0j}(t)\|_{M'}\\ &\le CD\bar{C}\sum_{j=0}^{\infty}\hat{\chi}_j < \infty,\end{aligned}$$

which shows the assertion.

(ii) Let $t,t' \in [0,1]$. From Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds) we obtain with some constant $C > 0$, for each $k = 1,...,d_\Theta$,

$$\begin{aligned}\|\nabla_{\theta_k} g(\tilde{Z}_0(t),\theta(t))\|_1 &\le C,\\ \|\nabla_{\theta_k} g(\tilde{Z}_0(t),\theta(t)) - \nabla_{\theta_k} g(\tilde{Z}_0(t),\theta(t'))\|_1 &\le C|\theta(t)-\theta(t')|_1.\end{aligned}$$

From Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds) we obtain for each $k = 1,...,d_\Theta$ (note that $rM \ge M'$ in Assumption 2.2):

$$\begin{aligned}&\|\nabla_{\theta_k} g(\tilde{Z}_0(t),\theta(t')) - \nabla_{\theta_k} g(\tilde{Z}_0(t'),\theta(t'))\|_1\\ &\le \bar{C}C\big(\|\tilde{Y}_0(t)-\tilde{Y}_0(t')\|_{M'} + \sum_{j=1}^{\infty}\chi_j\|\tilde{X}_0(t)-\tilde{X}_0(t')\|_{M'}\big)\\ &\le \bar{C}CC_B|t-t'|(1+|\chi|_1). \qquad (7.40)\end{aligned}$$

By Lipschitz continuity of θ, θ', the above results imply that the second summand in (7.38) fulfills the assertion,

$$\sup_{t \neq t'} \frac{\|\nabla_\theta g(\tilde{Z}_0(t), \theta(t))\theta'(t) - \nabla_\theta g(\tilde{Z}_0(t'), \theta(t'))\theta'(t')\|_1}{|t - t'|} < \infty.$$

It remains to show the same for the first summand in (7.38). By (7.39) and $\|\partial_t \tilde{Z}_{0j}(t) - \partial_t \tilde{Z}_{0j}(t')\|_{M'} \le C_B|t - t'|$ from Assumption 2.2(B3), we have

$$\|\partial_z g(\tilde{Z}_0(t), \theta(t))(\partial_t \tilde{Z}_0(t) - \partial_t \tilde{Z}_0(t'))\|_1 \le C\bar{C}C_B|\chi|_1|t - t'|. \tag{7.41}$$

Similar as in (7.40), we see by Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds) that

$$\|\partial_{z_j} g(\tilde{Z}_0(t), \theta(t)) - \partial_{z_j} g(\tilde{Z}_0(t'), \theta(t))\|_{M'/(M'-1)} \le \chi_j \bar{C}CC_B(1 + |\chi'|_1)|t - t'|. \tag{7.42}$$

Finally, by Lemma 7.1 (if Assumption 2.1 holds) or Lemma 7.3 (if Assumption 7.16 holds) and Lipschitz continuity of θ, we have

$$\|\partial_{z_j} g(\tilde{Z}_0(t'), \theta(t)) - \partial_{z_j} g(\tilde{Z}_0(t'), \theta(t'))\|_{M'/(M'-1)} \le \chi_j \bar{C}C|\theta(t) - \theta(t')|_1 = O(|t - t'|). \tag{7.43}$$

By Hölder's inequality, we conclude from (7.42) and (7.43) that

$$\|(\partial_z g(\tilde{Z}_0(t), \theta(t)) - \partial_z g(\tilde{Z}_0(t'), \theta(t')))\partial_t \tilde{Z}_0(t')\|_1 = O(|t - t'|),$$

which together with (7.41), finishes the proof. □

LEMMA 7.5. *Let $q \ge 1$. Suppose that Assumption 2.1(A5), (A7) hold with some $r \ge q$. Let $g \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$, where $\chi_i = O(i^{-(1+\gamma)})$. Then it holds that*

(i) $\sup_{t\in[0,1]} \delta_q^{\sup_\theta |g(\tilde{Z}(t),\theta)|}(j) = O(j^{-(1+\gamma)})$.

(ii) For $M_i(t, \eta, u) := \hat{K}_{b_n}(u - t)g(\tilde{Z}_i(u), \eta_1 + \eta_2(u - t)b_n^{-1})$, we have

$$\sup_{u\in[0,1]} \sup_{t,\eta} \delta_q^{M(t,\eta,u)}(j) = O(j^{-(1+\gamma)}), \qquad \sup_{u\in[0,1]} \delta_q^{\sup_{t,\eta} |M(t,\eta,u)|}(j) = O(j^{-(1+\gamma)}).$$

(iii) Let $d_u(t) = \theta(u) - \theta(t) - (u - t)\theta'(t)$ and $M_i^{(2)}(t, u) := \hat{K}_{b_n}(u - t)\{\int_0^1 g(\tilde{Z}_i(u), \theta(t) + sd_u(t))ds\} \cdot d_u(t)$. Then it holds for each component that

$$\sup_{u\in[0,1]} \delta_q^{M^{(2)}(t,u)}(j) = O(b_n^2 j^{-(1+\gamma)}), \qquad \sup_{u\in[0,1]} \delta_q^{\sup_t |M^{(2)}(t,u)|}(j) = O(b_n^2 j^{-(1+\gamma)}).$$

(*) If instead Assumption 7.16(A5'), (A7') hold with some $r > q$ and $g = \ell$ fulfills (7.94) for all $s > 0$ small enough,then the statements above remain valid.

PROOF. (i) Let $\tilde{Z}_j(t)^*$ be a coupled version of $\tilde{Z}_j(t)$ where ζ_0 is replaced by ζ_0^*. By Lemma 7.1 we obtain in case (2.12) that with some constant $\tilde{C} > 0$:

$$\begin{aligned}
&\delta_q^{\sup_\theta |g(\tilde{Z}(t),\theta)|}(j)\\
= \ & \| \sup_\theta |g(\tilde{Z}_j(t),\theta)| - \sup_\theta |g(\tilde{Z}_j(t)^*,\theta)| \|_q\\
\le \ & \| \sup_\theta |g(\tilde{Z}_j(t),\theta) - g(\tilde{Z}_j(t)^*,\theta)| \|_q\\
\le \ & \tilde{C}\Big(\|\tilde{Y}_j(t) - \tilde{Y}_j(t)^*\|_{qM} + \sum_{i=1}^{\infty} \chi_i \|\tilde{X}_{j-i+1}(t) - \tilde{X}_{j-i+1}(t)^*\|_{qM}\Big)\\
\le \ & \tilde{C}\Big(\delta_{qM}^{\tilde{Y}(t)}(j) + \sum_{i=1}^{\infty} \chi_i \delta_{qM}^{\tilde{X}(t)}(j-i+1)\Big),
\end{aligned} \tag{7.44}$$

and in case (2.13), similarly

$$\delta_q^{g(\tilde{Z}(t),\theta)}(j) \le \tilde{C} \sum_{i=1}^{\infty} \chi_i \delta_{qM}^{\tilde{X}(t)}(j-i+1). \tag{7.45}$$

In case (*), let $s > 0$ be such that $q(1+s) < r$. Then we have by Lemma 7.2:

$$\begin{aligned}
\delta_q^{g(\tilde{Z}(t),\theta)}(j) \ &\le \ \tilde{C} \sum_{i=0}^{\infty} \hat{\chi}_i^{(s)} \big(\|\tilde{Z}_i(t) - \tilde{Z}_i(t)^*\|_{qM} + \|\tilde{Z}_i(t) - \tilde{Z}_i(t)^*\|_{qM(1+s)}^s \big)\\
&\le \ \tilde{C} \sum_{i=0}^{\infty} \chi_i^{(s)} \big(\delta_{qM}^{\tilde{X}(t)}(j-i+1) + [\delta_{qM(1+s)}^{\tilde{X}(t)}(j-i+1)]^s \big).
\end{aligned} \tag{7.46}$$

Note that if two sequences a_i, b_i with $a_i = b_i = 0$ for $i < 0$ obey $a_i, b_i = O(i^{-(1+\gamma)})$ then the convolution $c_j = \sum_{i=1}^{\infty} a_i b_{j-i+1}$ still obeys $c_j = O(j^{-(1+\gamma)})$ due to

$$\begin{aligned}
|c_j| \ &\le \ \sum_{i=1, i \ge (j+1)/2}^{j+1} |a_i| \cdot |b_{j-i+1}| + \sum_{i=1, |j-i| \ge (j+1)/2}^{j+1} |a_i||b_{j-i+1}|\\
&\le \ \big(\frac{j+1}{2}\big)^{-(1+\gamma)} \sum_{i=1}^{j+1} |b_{j-i+1}| + \big(\frac{j+1}{2}\big)^{-(1+\gamma)} \sum_{i=1}^{j+1} |a_i| = O(j^{-(1+\gamma)}).
\end{aligned}$$

Together with Assumption (A7) and (7.44), (7.45) or (in case (*)) Assumption 7.16(A7') and (7.46), this shows $\sup_{t \in [0,1]} \delta_r^{g(\tilde{Z}(t),\theta)}(j) = O(j^{-(1+\gamma)})$.

The proof for (ii),(iii) is the same since

$$\begin{aligned}\big|\sup_{t,\eta}|M_i(t,\eta,u)| - \sup_{t,\eta}|M_i(t,\eta,u)^*|\big| &\leq \sup_{t,\eta}|M_i(t,\eta,u) - M_i(t,\eta,u)^*| \\ &\leq |\hat{K}|_\infty \sup_\theta |g(\tilde{Z}_i(u),\theta) - g(\tilde{Z}_i(u)^*,\theta)|\end{aligned}$$

and (since $|d_u(t)|_\infty \leq \sup_s|\theta''(s)|_\infty \cdot b_n^2$ if $|t-u| \leq b_n$), for each $l = 1,\ldots,k$,

$$\begin{aligned}&\big|\sup_t|\tilde{M}_i^{(2)}(t,u)_l| - \sup_t|\tilde{M}_i^{(2)}(t,u)_l^*|\big| \leq \sup_t|M_i^{(2)}(t,u)_l - M_i^{(2)}(t,u)_l^*| \\ \leq\ & |\hat{K}|_\infty \sup_s|\theta''(s)|_\infty b_n^2 \\ &\quad \times \sup_t \int_0^1 |g(\tilde{Z}_i(u),\theta(t)+sd_u(t)) - g(\tilde{Z}_i(u)^*,\theta(t)+sd_u(t))|ds \\ \leq\ & |\hat{K}|_\infty \sup_s|\theta''(s)|_\infty b_n^2 \sup_{\theta\in\Theta}|g(\tilde{Z}_i(u),\theta) - g(\tilde{Z}_i(u)^*,\theta)|.\end{aligned}$$

□

LEMMA 7.6 (for tvGARCH). *Let $q \geq 1$. Suppose that Assumption 7.16(A5'), (A7') hold with some $r > q$. For $s > 0$, let $\chi^{(s)} = (\chi_i^{(s)})_{i\in\mathbb{N}}$ be such that $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Let g be such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(x,\tilde{\theta},y),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M_y,M_x,\chi^{(s)},\bar{C}^{(s)})$ for all $s > 0$ small enough. Then*

(i) $\sup_{t\in[0,1]} \delta_q^{\sup_{|\theta-\theta(t)|_1<\iota}|g(\tilde{Z}(t),\theta)|}(j) = O(j^{-(1+\gamma)})$.

(ii) For n large enough,

$$\sup_{u\in[0,1]}\sup_{t,|\eta-\eta_{b_n}(t)|_1<\iota/2}\delta_q^{M(t,\eta,u)}(j) = O(j^{-(1+\gamma)}), \qquad \sup_{u\in[0,1]}\delta_q^{\sup_{t,|\eta-\eta_{b_n}(t)|_1<\iota/2}|M(t,\eta,u)|}(j) = O(j^{-(1+\gamma)}).$$

(iii) For n large enough, $\sup_{u\in[0,1]}\delta_q^{M^{(2)}(t,u)}(j) = O(b_n^2 j^{-(1+\gamma)})$, and $\sup_{u\in[0,1]}\delta_q^{\sup_t|M^{(2)}(t,u)|}(j) = O(b_n^2 j^{-(1+\gamma)})$.

PROOF OF LEMMA 7.6. (i) Let $\tilde{Z}_j(t)^*$ be a coupled version of $\tilde{Z}_j(t)$ where ζ_0 is replaced by ζ_0^*. By Lemma 7.3 we obtain that with some constant $\tilde{C} > 0$:

$$\begin{aligned}
&\delta_q^{\sup_{|\theta-\theta(t)|_1<\iota}|g(\tilde{Z}(t),\theta)|}(j)\\
\leq\ & \| \sup_{|\theta-\theta(t)|_1<\iota} |\tilde{g}_{\theta(t)}(\zeta_j, \tilde{X}_j(t), \theta) - \tilde{g}_{\theta(t)}(\zeta_j, \tilde{X}_j(t)^*, \theta)|\|_q\\
\leq\ & \tilde{C}\sum_{i=1}^{\infty}\chi_i\|\tilde{X}_{j-i+1}(t) - \tilde{X}_{j-i+1}(t)^*\|_{qM}\\
\leq\ & \tilde{C}\sum_{i=1}^{\infty}\chi_i\delta_{qM}^{\tilde{X}(t)}(j-i+1).
\end{aligned}$$

The result now follows as in the proof of Lemma 7.5(i) with Assumption 7.16(A7').

(ii) We have for n large enough that

$$|\eta-\eta_{b_n}(t)|_1 = |\eta_1-\theta(t)|_1+|\eta_2-b_n\theta'(t)|_1 < \iota/2 \quad \text{implies} \quad |(\eta_1+\eta_2(u-t)b_n^{-1})-\theta(t)|_1 \leq |\eta_1-\theta(t)|_1+|\eta_2|_1 < \iota$$

and $|\theta - \theta(t)|_1 < \iota$, $|u-t| \leq b_n$ implies $|\theta-\theta(u)|_1 < \iota$ due to uniform continuity of $\theta(\cdot)$. Therefore, we have for n large enough:

$$\begin{aligned}
&\Big|\sup_{t,|\eta-\eta_{b_n}(t)|_1<\iota/2}|M_i(t,\eta,u)| - \sup_{t,|\eta-\eta_{b_n}(t)|_1<\iota/2}|M_i(t,\eta,u)^*|\Big|\\
\leq\ & \sup_{t,|\eta-\eta_{b_n}(t)|_1<\iota/2}|\hat{K}_{b_n}(u-t)|\cdot|g(\tilde{Z}_i(u),\eta_1+\eta_2(u-t)b_n^{-1}) - g(\tilde{Z}_i(u),\eta_1+\eta_2(u-t)b_n^{-1})|\\
\leq\ & \sup_{t,|\theta-\theta(t)|_1<\iota}|\hat{K}_{b_n}(u-t)|\cdot|\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta) - \tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u)^*,\theta)|\\
\leq\ & |\hat{K}|_\infty\cdot\sup_{|\theta-\theta(u)|_1<\iota}|\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta) - \tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u)^*,\theta)|.
\end{aligned}$$

The rest works as in (i).

(iii) For n large enough, it holds that $|u-t| \leq b_n$ implies that $\sup_{s\in[0,1]}|\theta(t)+sd_u(t)-\theta(u)| < \iota$ due to uniform continuity of $\theta(\cdot)$. Thus

$$\begin{aligned}
&\big|\sup_t|M_i^{(2)}(t,u)| - \sup_t|M_i^{(2)}(t,u)^*|\big|\\
\leq\ & |\hat{K}|_\infty\sup_s|\theta''(s)|_\infty b_n^2\sup_{|\theta-\theta(u)|<\iota}|\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta) - \tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u)^*,\theta)|.
\end{aligned}$$

The rest works as in (i). □

LEMMA 7.7 (Lipschitz properties of $\hat{G}_n$). *Let* $s \geq 0$.

(i) *Let $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$. Let Assumption 2.1(A5) hold with $r \geq 1+s$. Then there exists some constant $\tilde{C} > 0$ such that*

$$\sup_{t\in[0,1]} \Big\| \sup_{\eta\neq\eta'} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')|}{|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C},$$

and

$$\Big\| \sup_{t\neq t'} \sup_{\eta\neq\eta'} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')|}{|t-t'| + |\eta-\eta'|_1} \Big\|_1 \leq \tilde{C} b_n^{-2},$$

(ii) *(for tvGARCH) Let g be such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(x,\tilde{\theta},y),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$ with $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Let Assumption 7.16(A5') hold with $r \geq 1+s$ and let $\theta(\cdot)$ be continuous. Then there exists some constant $\tilde{C}^{(s)} > 0$ such that*

$$\sup_{t\in[0,1]} \Big\| \sup_{\substack{\eta\neq\eta' \\ |\eta-\eta_{b_n}(t)|_1<\iota/2, |\eta'-\eta_{b_n}(t)|_1<\iota/2}} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')|}{|\eta-\eta'|_1} \Big\|_1 \leq \tilde{C}^{(s)},$$

and

$$\Big\| \sup_{t\neq t'} \sup_{\substack{\eta\neq\eta' \\ |\eta-\eta_{b_n}(t)|_1<\iota/2, |\eta'-\eta_{b_n}(t')|_1<\iota/2}} \frac{|\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')|}{|t-t'| + |\eta-\eta'|_1} \Big\|_1 \leq \tilde{C}^{(s)} b_n^{-2},$$

PROOF OF LEMMA 7.7. By Lemma 7.1(i), $\sup_{t\in[0,1]} \|R_{M_y,M_x}(\tilde{Z}_0(t))^{1+s}\|_1 < \infty$. This is needed several times in the following.
(i) Since $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$ and $|i/n - t| \leq b_n$ inside the sum, it holds that
(7.47)

$$|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| \leq \bar{C}|\eta-\eta'|_1 \cdot (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n-t)| \{R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s} + \|R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s}\|_1\}.$$

Furthermore, $(nb_n)^{-1}\sum_{i=1}^n |\hat{K}_{b_n}(i/n-t)| \leq |\hat{K}|_\infty$. This yields the assertion.
Since $g \in \mathcal{H}_s(M_y, M_x, \chi, \bar{C})$, we have with some constant $\tilde{C} > 0$:

$$\begin{aligned}
& |\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')| \\
\leq\ & (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n-t) - \hat{K}_{b_n}(i/n-t')| \cdot \sup_\theta \{|g(\tilde{Z}_i(i/n),\theta)| + \|g(\tilde{Z}_i(i/n),\theta)\|_1\} \\
& + (nb_n)^{-1} \sum_{i=1}^n |\hat{K}_{b_n}(i/n-t')| \cdot |g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n-t)b_n^{-1}) - g(\tilde{Z}_i(i/n), \eta_1' + \eta_2'(i/n-t')b_n^{-1}) \\
\leq\ & \big[b_n^{-2} L_{\hat{K}} |t-t'| + b_n^{-1}|\hat{K}|_\infty \{|\eta-\eta'|_1 + |\eta_2| \cdot |t-t'| b_n^{-1}\big] \cdot \frac{1}{n}\sum_{i=1}^n \{R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s} \\
& \qquad + \|R_{M_y,M_x}(\tilde{Z}_i(i/n))^{1+s}\|_1\}.
\end{aligned}$$

Since $\sup_{\eta\in E_n}|\eta_2|_1 < \infty$ is compact, this gives the result.
(ii) We now have

$$\begin{aligned}
& |\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| \\
\le\ & (nb_n)^{-1}\sum_{i=1}^n |\hat{K}_{b_n}(i/n-t)|\cdot\big|\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}) \\
& \qquad -\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1'+\eta_2'(i/n-t)b_n^{-1})\big|.
\end{aligned}$$

Here, $|\eta-\eta_{b_n}(t)| < \iota/2$ implies $|(\eta_1+\eta_2(i/n-t)b_n^{-1})-\theta(t)| < \iota$ for n large enough. Since $\theta(\cdot)$ is uniformly continuous, $|\theta-\theta(t)|_1 < \iota$, $|i/n-t| \le b_n$ imply $|\theta-\theta(i/n)|_1 < \iota$ for n large enough. Since $\tilde{g}_{\tilde{\theta}} \in \mathcal{H}_{s,\iota}^{mult}(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, we obtain

$$\begin{aligned}
|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| \le\ & \bar{C}^{(s)}|\eta-\eta'|_1 (nb_n)^{-1}\sum_{i=1}^n |\hat{K}_{b_n}(i/n-t)|\cdot\{R_{M,M}(1,\tilde{X}_i(i/n))^{1+s}(1+|\zeta_i|^M)^{1+s} \\
& +\|R_{M,M}(1,\tilde{X}_i(i/n))^{1+s}(1+|\zeta_i|^M)^{1+s}\|_1\},
\end{aligned}$$

giving the result.
We have

$$\begin{aligned}
& |\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')| \\
\le\ & (nb_n)^{-1}\sum_{i=1}^n |\hat{K}_{b_n}(i/n-t) - \hat{K}_{b_n}(i/n-t')| \\
& \quad\times \sup_{|\eta-\eta_{b_n}(t)|<\iota/2}\{|\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1})| + \|\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1})\|_1\} \\
& +(nb_n)^{-1}\sum_{i=1}^n |\hat{K}_{b_n}(i/n-t')|\cdot|\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}) \\
& \qquad -\tilde{g}_{\theta(i/n)}(\zeta_i,\tilde{X}_i(i/n),\eta_1'+\eta_2'(i/n-t')b_n^{-1}).
\end{aligned}$$

The same argumentation as before allows us to use the Lipschitz properties of $\tilde{g}_{\theta(i/n)}$ w.r.t. θ, giving the result. □

For the proof of the following lemma, we will make use of the adjusted dependence measure $\|\cdot\|_{q,\alpha}$ which is defined as follows (cf. [61]): For some zero-mean random variable Z, let $\|Z\|_{q,\alpha} := \sup_{m\ge 0}(m+1)^\alpha \Delta_q^Z(m)$.

LEMMA 7.8. *Let $\gamma > 1$. For $s \ge 0$, let $\chi_i^{(s)} = (\chi_i^{(s)})_{i\in\mathbb{N}}$ be a sequence with $\chi_i^{(s)} = O(i^{-(1+\gamma)})$. Recall the notation from (7.1). Assume that either (in the case (a)) Assumption*

2.1(A5), (A6), (A7) or (in the cases (b),(c)) Assumption 7.16(A5'), (A6'), (A7') hold with some r specified below.

(i) Let $r \geq 1+\varsigma$, $\varsigma \geq 0$ and assume either that $\varsigma = 0$ and $g \in \mathcal{H}_0(M_y, M_x, \chi^{(0)}, \bar{C}^{(0)})$ or $\varsigma > 0$ and for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$. Then

$$\| \sup_{t\in(0,1)} \sup_{\eta \in E_n} |\hat{G}_n(t,\eta) - G_n^c(t,\eta)| \|_1 = O((nb_n)^{-1}).$$

(ii) Fix $t \in [0,1]$ and assume that $nb_n \to \infty$. Let $r \geq 1+\varsigma$, $\varsigma > 0$.

(a) If for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{\eta\in E_n} |\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1).$$

(b) If for all $s > 0$ small enough, $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(y,x,\tilde{\theta}),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{|\eta - \eta_{b_n}(t)|<\iota} |\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1) \qquad \text{if} \quad b_n \to 0.$$

(c) If for all $s > 0$ small enough, g fulfills (7.94) and $g \in \mathcal{H}_s(2M_y, 2M_x, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{\eta\in E_n} |\hat{G}_n(t,\eta)| = o_{\mathbb{P}}(1).$$

(iii) Let $r \geq 2+\varsigma$, $\varsigma > 0$. Define $\beta_n = \log(n)^{1/2}(nb_n)^{-1/2}b_n^{-1/2}$.

(a) If for all $s > 0$ small enough, $g \in \mathcal{H}_s(M_y, M_x, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{t\in(0,1)} \sup_{\eta\in E_n} |\hat{G}_n(t,\eta)| = O_{\mathbb{P}}(\beta_n).$$

(b) If g is such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(y,x,\tilde{\theta}),x,\theta)$ and for all $s > 0$ small enough, $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{t\in(0,1)} \sup_{|\eta-\eta_{b_n}(t)|_1<\iota} |\hat{G}(t,\eta)| = O_{\mathbb{P}}(\beta_n).$$

(c) If for all $s > 0$ small enough, g fulfills (7.94) and $g \in \mathcal{H}_s(2M_y, 2M_x, \chi^{(s)}, \bar{C}^{(s)})$, then

$$\sup_{t\in(0,1)} \sup_{\eta\in E_n} |\hat{G}(t,\eta)| = O_{\mathbb{P}}(\beta_n).$$

PROOF OF LEMMA 7.8. We abbreviate $\chi = \chi^{(s)}$ and $\bar{C} = \bar{C}^{(s)}$.

(i) By Lemma 7.1(i),(ii) and by Assumption 2.1(A6), we obtain (independent of (2.12) or (2.13)) that for some $C > 0$:

$$\|\sup_{\theta\in\Theta}|g(Z_i,\theta) - g(Z_i^c,\theta)|\|_1 \quad\le\quad C\sum_{j=0}^{\infty}\hat{\chi}_j\|Z_{ij} - Z_{ij}^c\|_M \le 2C\sum_{j=i}^{\infty}\chi_j\|Z_{ij}\|_M \le 2CD\sum_{j=i}^{\infty}\chi_j.$$

Similarly, we have in the case (2.12) that

$$\begin{aligned}\|\sup_{\theta\in\Theta}|g(Z_i,\theta) - g(\tilde{Z}_i(i/n),\theta)|\|_1 \quad&\le\quad C\Big(\|Y_i - \tilde{Y}_i(i/n)\|_M + \sum_{j=1}^{\infty}\chi_j\|X_{ij} - \tilde{X}_{ij}(i/n)\|_M\Big)\\ &\le\quad CC_A|\chi|_1 n^{-1},\end{aligned}$$

while in the case (2.13) there exists $C_2 > 0$ such that

$$\|\sup_{\theta\in\Theta}|g(Z_i,\theta) - g(\tilde{Z}_i(i/n),\theta)|\|_1 \quad\le\quad C_2\sum_{j=1}^{\infty}\chi_j\|X_{ij} - \tilde{X}_{ij}(i/n)\|_M \le C_2C_A|\chi|_1 n^{-1}.$$

Thus

$$\begin{aligned}&\|\sup_{t\in(0,1)}\sup_{\eta\in E_n}|G_n(t,\eta) - G_n^c(t,\eta)|\|_1\\ \le\quad& |K|_\infty(nb_n)^{-1}\sum_{i=1}^{n}\|\sup_{\theta\in\Theta}|g(Z_i,\theta) - g(Z_i^c,\theta)|\|_1\\ \le\quad& 2CD|K|_\infty(nb_n)^{-1}\sum_{i=1}^{n}\sum_{j=i}^{\infty}\chi_j + |K|_\infty(C\vee C_2)|\chi|_1(nb_n)^{-1}.\end{aligned}$$

Since $\chi_j = O(j^{-1+\gamma})$, it holds that $\sum_{i=1}^{n}\sum_{j=i}^{\infty}\chi_j = O(1)$ and the assertion is proved. The proofs under Assumption 7.16 are similar in view of Lemma 7.2.

(ii) (a) Fix $Q > 0$. Let $\kappa > 0$. Let $E_n^{(\kappa)}$ be a discretization of E_n such that for each $\eta \in E_n$ one can find $\eta' \in E_n^{(\kappa)}$ with $|\eta - \eta'|_1 \le \kappa$. Note that $\#E_n^{(\kappa)}$ does not need to depend on n. Then

$$\begin{aligned}\mathbb{P}\big(\sup_{\eta\in E_n}|\hat{G}_n(t,\eta)| > Q\big) \quad&\le\quad \#E_n^{(\kappa)}\sup_{\eta\in E_n}\mathbb{P}\big(|\hat{G}_n(t,\eta)| > Q/2\big)\\ &\quad +\mathbb{P}\big(\sup_{|\eta-\eta'|_1\le\kappa}|\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| > Q/2\big).\end{aligned}\tag{7.48}$$

By Markov's inequality, we have for $0 \le s \le \varsigma$,

$$\mathbb{P}\big(|\hat{G}_n(t,\eta)| > Q/2\big) \le \frac{\|\hat{G}_n(t,\eta)\|_{1+s}^{1+s}}{(Q/2)^{1+s}}.$$

Using Burkholder's moment inequality (cf. [8]) and Lemma 7.5(i) applied for $q = 1 + s$, $s > 0$ small enough, the computation

$$\begin{aligned}
(7.49)\ & \|\hat{G}_n(t,\eta)\|_{1+s} \\
\leq\ & (nb_n)^{-1} \sum_{l=0}^{\infty} \Big\| \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t) P_{i-l} g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) \Big\|_{1+s} \\
\leq\ & s^{-1}(nb_n)^{-1} \sum_{l=0}^{\infty} \Big(\Big\| \sum_{i=1}^{n} \hat{K}_{b_n}(i/n - t)^2 P_{i-l}^2 g(\tilde{Z}_i(i/n), \eta_1 + \eta_2(i/n - t)b_n^{-1}) \Big\|_{(1+s)/2}^{(1+s)/2} \Big)^{1/(1+s)} \\
\leq\ & s^{-1}(nb_n)^{-s/(1+s)} |\hat{K}|_\infty \sum_{l=0}^{\infty} \sup_{t\in[0,1]} \delta_{1+s}^{\sup_{\theta\in\Theta}|g(\tilde{Z}(t),\theta)|}(l) = O((nb_n)^{-s/(1+s)}),
\end{aligned}$$

shows that the first summand in (7.48) tends to zero. For the second summand, Lemma 7.7(i) implies

$$\mathbb{P}(\sup_{|\eta-\eta'|_1 \leq \kappa} |\hat{G}_n(t,\eta) - \hat{G}_n(t,\eta')| > Q/2) \leq \frac{2\tilde{C}\kappa}{Q},$$

which can be made arbitrary small by choosing ι small enough. So we have shown that (7.48) tends to zero for $n \to \infty$.

(b) The proof is similar to (a) by using 7.7(ii) and Lemma 7.6(i) instead of Lemma 7.7(i) and Lemma 7.5(i).

(c) The proof is similar to (a) by using Lemma 7.5(i)(*) instead of Lemma 7.5(i).

(iii) (a) We use a chaining argument. Let $r = n^3$ and let $E_{n,r}$ be a discretization of E_n such that for each $\eta \in E_n$ one can find $\eta' \in E_{n,r}$ with $|\eta - \eta'| \leq r^{-1}$. Define $\mathcal{T}_{n,r} := \{i/r : i = 1, \dots, r\}$ as a discretization of $(0,1)$. Then $\#(E_{n,r} \times \mathcal{T}_{n,r}) = O(r^{2d_\Theta+1})$. For some constant $Q > 0$, we have

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{\eta\in E_n, t\in(0,1)} |\hat{G}_n(t,\eta)| > Q\beta_n \Big) \\
\leq\ & \mathbb{P}\Big(\sup_{\eta\in E_{n,r}, t\in\mathcal{T}_{n,r}} |\hat{G}_n(t,\eta)| > Q\beta_n/2 \Big) \\
& + \mathbb{P}\Big(\sup_{|\eta-\eta'|\leq r^{-1}, |t-t'|\leq r^{-1}} |\hat{G}_n(t,\eta) - \hat{G}_n(t',\eta')| > Q\beta_n/2 \Big). \qquad (7.50)
\end{aligned}$$

Let $\alpha = 1/2$. Let $M_i(t,\eta,u) := \hat{K}_{b_n}(u - t) g(\tilde{Z}_i(u), \eta_1 + \eta_2(u - t)b_n^{-1})$. By Lemma 7.5(ii) applied with $q = 2 + s$, $s > 0$ small enough, we have $\sup_u \Delta_{2+s}^{\sup_{t,\eta}|M(t,\eta,u)|}(k) = O(k^{-(1+\gamma)})$. Thus

$$W_{2+s,\alpha} := \sup_{u\in[0,1]} \| \sup_{t,\eta} |M_i(t,\eta,u)| \|_{2+s,\alpha} = \sup_{m\geq 0} (m+1)^\alpha \sup_{u\in[0,1]} \sup_{t,\eta} \Delta_{2+s}^{\sup_{t,\eta}|M(t,\eta,u)|}(m) < \infty.$$

(independent of n) and

$$W_{2,\alpha} \quad := \quad \sup_{u\in[0,1]}\sup_{t,\eta}\|M_i(t,\eta,u)\|_{2,\alpha} = \sup_{m\geq 0}(m+1)^{\alpha}\sup_{u\in[0,1]}\sup_{t,\eta}\Delta_2^{M(t,\eta,u)}(m) < \infty$$

(independent of n). Note that $l = 1\wedge\log\#(E_{n,r}\times\mathcal{T}_{n,r}) \leq 3(2d_\Theta+1)\log(n)$ and $Q\beta_n(nb_n) = Qn^{1/2}\log(n)^{1/2} \geq \sqrt{nl}W_{2,\alpha} + n^{1/(2+s)}l^{3/2}W_{2+s,\alpha} \gtrsim n^{1/2}\log(n)^{1/2} + n^{1/(2+s)}\log(n)^{3/2}$ for Q large enough. By applying Theorem 6.2 of [61] (the proof therein also works for the uniform functional dependence measure) with $q = 2+s$ and $\alpha = 1/2$ to $(M_i(t,\eta,i/n))_{t\in\mathcal{T}_{n,r},\eta\in E_{n,r}}$, we have with some constant $C_\alpha > 0$:

$$\begin{aligned}
&\mathbb{P}\big(\sup_{\eta'\in E_{n,r},t'\in\mathcal{T}_{n,r}}|\hat{G}_n(t',\eta')| \geq Q\beta_n/2\big)\\
\leq\ & \frac{C_\alpha n\cdot l^{1+s/2}W_{2+s,\alpha}^{2+s}}{(Q/2)^{2+s}(\delta_n(nb_n))^{2+s}} + C_\alpha\exp\Big(-\frac{C_\alpha(Q/2)^2(\beta_n(nb_n))^2}{nW_{2,\alpha}^2}\Big)\\
\lesssim\ & n^{-s/2} + \exp\big(-\frac{(nb_n)b_n^{-1}\log(n)}{n}\big)\\
\to\ & 0. \qquad (7.51)
\end{aligned}$$

By Markov's inequality and Lemma 7.7(i),

$$\mathbb{P}\big(\sup_{|\eta-\eta'|_1\leq r^{-1},|t-t'|\leq r^{-1}}|\hat{G}_n(t,\eta)-\hat{G}_n(t',\eta')| \geq C\beta_n/2\big) = O\Big(\frac{b_n^{-2}r^{-1}}{\beta_n}\Big). \qquad (7.52)$$

We have $b_n^{-2}r^{-1}\beta_n^{-1} = b_n^{-2}n^{-3}(nb_n)^{1/2}b_n^{1/2}\log(n)^{-1/2} \to 0$. Inserting (7.51) and (7.52) into (7.50), we obtain the result.

(b) The proof is similar to (a) by using 7.7(ii) and Lemma 7.6(ii) instead of Lemma 7.7(i) and Lemma 7.5(ii).

(c) The proof is similar to (a) by using 7.7(ii)(*) instead of Lemma 7.7(ii). □

LEMMA 7.9. *For $g:\mathbb{R}\times\mathbb{R}^{\mathbb{N}}\times\Theta\to\mathbb{R}$. Let*

$$\hat{B}_n(t,\eta) = (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)g(\tilde{Z}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}).$$

(a) If Assumption 2.1(A5) is fulfilled with $r\geq 1+s$, $s\geq 0$ and $g\in\mathcal{H}_s(M_y,M_x,\chi,\bar{C})$, then

$$\sup_{t\in(0,1)}\sup_{\eta\in E_n}|\mathbb{E}\hat{B}_n(t,\eta) - \int_{-t/b_n}^{(1-t)/b_n}\hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t),\eta_1+\eta_2 x)dx| = O((nb_n)^{-1}+b_n).$$

(b) If Assumption 7.16(A5') is fulfilled with $r \geq 1+s$ and g is such that $\tilde{g}_{\tilde{\theta}}(y,x,\theta) := g(F(y,x,\tilde{\theta}),x,\theta)$ fulfills $\tilde{g} \in \mathcal{H}_{s,\iota}^{mult}(M,\chi,\bar{C})$, then

$$\sup_{t\in(0,1)}\sup_{|\eta-\eta_{b_n}(t)|<\iota}|\mathbb{E}\hat{B}_n(t,\eta) - \int_{-t/b_n}^{(1-t)/b_n}\hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t),\eta_1+\eta_2 x)dx| = O((nb_n)^{-1}+b_n).$$

If the supremum is taken over $t \in \mathcal{T}_n$ instead of $t \in (0,1)$, then $\int_{-t/b_n}^{(1-t)/b_n}$ can be replaced by $\int_{-1}^{1}$.

Proof of Lemma 7.9. (a) Let $\tilde{B}_n(t,\eta) := (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)g(\tilde{Z}_i(t),\eta_1+\eta_2(i/n-t)b_n^{-1})$. By Lemma 7.1(i), we have with some constant $\tilde{C}>0$ that either in the case of (2.12),

$$\begin{aligned}
&\|g(\tilde{Z}_0(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}) - g(\tilde{Z}_0(t),\eta_1+\eta_2(i/n-t)b_n^{-1})\|_1\\
\leq\ &\tilde{C}\Big(\|\tilde{Y}_0(i/n)-\tilde{Y}_0(t)\|_M + \sum_{i=1}^{\infty}\chi_i\|\tilde{X}_{-i}(i/n)-\tilde{X}_{-i}(t)\|_M\Big) \leq \tilde{C}C_B(1+|\chi|_1)b_n
\end{aligned}$$

or in the case of (2.13),

$$\begin{aligned}
&\|g(\tilde{Z}_0(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}) - g(\tilde{Z}_0(t),\eta_1+\eta_2(i/n-t)b_n^{-1})\|_1\\
\leq\ &\tilde{C}\sum_{i=1}^{\infty}\chi_i\|\tilde{X}_{-i}(i/n)-\tilde{X}_{-i}(t)\|_M \leq \tilde{C}C_B|\chi|_1 b_n.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|\hat{B}_n(t,\eta)-\tilde{B}_n(t,\eta)\|_1\\
\leq\ &(nb_n)^{-1}\sum_{i=1}^n|\hat{K}_{b_n}(i/n-t)|\\
&\qquad\times\|g(\tilde{Z}_i(i/n),\eta_1+\eta_2(i/n-t)b_n^{-1}) - g(\tilde{Z}_i(t),\eta_1+\eta_2(i/n-t)b_n^{-1})\|_1\\
\leq\ &\tilde{C}|\hat{K}|_\infty C_B(1+|\chi|_1)b_n.
\end{aligned}$$

Since $\hat{K}$ is of bounded variation and $\theta \mapsto \mathbb{E}g(\tilde{Z}_0(t),\theta)$ is Lipschitz continuous due to $g \in \mathcal{H}(M_y,M_x,\chi,\bar{C})$ and Lemma 7.1, a Riemannian sum argument yields

$$\begin{aligned}
\tilde{B}_n(t,\eta) &= (nb_n)^{-1}\sum_{i=1}^n\hat{K}_{b_n}(i/n-t)\mathbb{E}g(\tilde{Z}_0(t),\eta_1+\eta_2(i/n-t)b_n^{-1})\\
&= \int_{-t/b_n}^{(1-t)/b_n}\hat{K}(x)\mathbb{E}g(\tilde{Z}_0(t),\eta_1+\eta_2 x)dx + O((nb_n)^{-1}),
\end{aligned}$$

uniformly in $t\in(0,1)$, $\eta\in E_n$.

(b) The proof is the same by using Lemma 7.3 with $q=1$ instead of Lemma 7.1. □

LEMMA 7.10. *Let $\eta_{b_n}(t) = (\theta(t)^\top, b_n\theta'(t)^\top)^\top$. Let Assumption 2.1 hold with $r = 1$ or let Assumption 7.16 hold with $r = 2 + \varsigma$, $\varsigma > 0$.*

(i) Then uniformly in $t \in \mathcal{T}_n$,

$$\mathbb{E}\nabla_{\eta_1}\hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) = b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) + O(b_n^3 + (nb_n)^{-1}). \tag{7.53}$$

Furthermore, it holds uniformly in $t \in (0, 1)$ that

$$\mathbb{E}\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) = \frac{b_n^2}{2} \int_{-t/b_n}^{(1-t)/b_n} K(x) \begin{pmatrix} x^2 \\ x^3 \end{pmatrix} dx \otimes [V(t)\theta''(t)] + O(b_n^3 + (nb_n)^{-1}). \tag{7.54}$$

(ii) If Assumptions 2.1, 2.2 or Assumptions 7.16, 7.17 hold with the r specified above, then uniformly in $t \in \mathcal{T}_n$,

$$\mathbb{E}\nabla_{\eta_2}\hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) = b_n^3 \frac{\mu_{K,4}}{2} V(t) bias(t) + O(b_n^4 + (nb_n)^{-1}),$$

where $bias(t) = \frac{1}{3}\theta^{(3)}(t) + V(t)^{-1}\mathbb{E}[\partial_t \nabla^2_\theta \ell(\tilde{Z}_0(t), \theta(t))] \cdot \theta''(t)$, and the term $O(b_n^3)$ in (7.53) can be replaced by $O(b_n^4)$.

PROOF OF LEMMA 7.10. (i) Let $U_{i,n}(t) = (K_{b_n}(i/n - t), K_{b_n}(i/n - t)(i/n - t)b_n^{-1})^\top$. By a Taylor expansion of $\theta(i/n)$ around t, we have

$$\theta(i/n) = \theta(t) + \theta'(t)(i/n - t) + r_n(t),$$

where $r_n(t) = \theta''(t)\frac{(i/n-t)^2}{2} + \theta'''(\tilde{t})\frac{(i/n-t)^3}{6}$ and $\tilde{t}$ is between t and i/n. We conclude that

$$\begin{aligned} &\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) - (nb_n)^{-1} \sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) \\ &= (nb_n)^{-1} \sum_{i=1}^n U_{i,n}(t) \otimes \Big\{ \int_0^1 \nabla^2_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n) + s r_n(t)) ds \cdot r_n(t) \Big\}. \end{aligned} \tag{7.55}$$

Using $\nabla^2_\theta \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ (if Assumption 2.1 holds) or $\nabla^2_\theta \ell \in \mathcal{H}_s(2M_y, 2M_x, \chi, \bar{C})$ with $s > 0$ small enough (if Assumption 7.16 holds), we obtain with Lemma 7.1 for $|i/n - t| \le b_n$:

$$\|\nabla^2_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n) + s r_n(t)) - \nabla^2_\theta \ell(\tilde{Z}_i(t), \theta(t))\|_1 = O(b_n + n^{-1}). \tag{7.56}$$

Using (7.55), $\mathbb{E}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) = 0$ (by Assumption 2.1(A1),(A3) or Assumption 7.16(A1'), (A3')) and (7.56), we obtain

$$\begin{aligned}
&\mathbb{E}\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) \\
&= (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \big\{\mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_i(t), \theta(t)) \cdot \theta''(t)\frac{(i/n-t)^2}{2}\big\} + O(b_n^3 + n^{-1}) \\
&= \begin{pmatrix} b_n^2\frac{\mu_{K,2}}{2}V(t)\theta''(t) \\ 0 \end{pmatrix} + O(b_n^3 + n^{-1} + (nb_n)^{-1}), \qquad (7.57)
\end{aligned}$$

which shows (7.53).

(7.54) follows by a more careful examination of the above Riemannian sum: Under Assumption 2.2, we have $r_n(t) = \theta''(t)\frac{(i/n-t)^2}{2} + \theta^{(3)}(t)\frac{(i/n-t)^3}{6} + \theta^{(4)}(\tilde{t})\frac{(i/n-t)^4}{24}$, where $\tilde{t}$ is between t and i/n. We now use a more precise Taylor argument as in (7.55). We have

$$\begin{aligned}
&\nabla_\eta \hat{L}^\circ_{n,b_n}(t, \eta_{b_n}(t)) - (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) \\
&= (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n)) r_n(t) \qquad (7.58) \\
&\quad + (nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \big\{\int_0^1 \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n) + s r_n(t)) \\
&\qquad\qquad - \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n)) ds \cdot r_n(t)\big\}.
\end{aligned}$$

Since $\nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$ (if Assumption 2.1 holds) or $\nabla_\theta^2 \ell \in \mathcal{H}_s(2M_y, 2M_x, \chi, \bar{C})$ for $s > 0$ small enough (if Assumption 7.16 holds), we have by Lemma 7.1:

$$\big\|\nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n) + s r_n(t)) - \nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta(i/n))\big\|_1 = O(r_n(t)) = O(|i/n - t|^2). \qquad (7.59)$$

This shows that the expectation of the second summand in (7.58) is $O(b_n^4)$. We now discuss the first term in (7.58). Put $v_i(t) := \nabla_\theta^2 \ell(\tilde{Z}_i(t), \theta(t))$. By Assumption 2.2, $t \mapsto v_i(t)$ is continuously differentiable. By Taylor's expansion, $v_i(i/n) = v_i(t) + (i/n - t)\partial_t v_i(t) + (i/n - t)\int_0^1 \partial_t v_i(t + s(i/n - t)) - \partial_t v_i(t) ds$. We have

$$\mathbb{E}\big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes v_i(t) r_n(t)\big] = \begin{pmatrix} b_n^2\frac{\mu_{K,2}}{2}V(t)\theta''(t) \\ b_n^3\frac{\mu_{K,4}}{6}V(t)\theta^{(3)}(t) \end{pmatrix} + O(n^{-1} + b_n^4), \qquad (7.60)$$

since K has bounded variation and $\int K(x)x^3 dx = 0$ by symmetry. Similarly,

$$\begin{aligned}
&\mathbb{E}\big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t)\otimes \partial_t v_i(t) r_n(t)\big] \\
&= \begin{pmatrix} 0 \\ b_n^3\frac{\mu_{K,4}}{2}\mathbb{E}[\partial_t \nabla_\theta^2 \ell(\tilde{Z}_0(t),\theta(t))]\theta''(t)\end{pmatrix} + O(n^{-1}+b_n^4). \qquad (7.61)
\end{aligned}$$

Finally, Lemma 7.4 applied to $g = \nabla_\theta^2 \ell$ (use Assumption 2.2 or 7.17) yields:

$$\|\partial_t v_i(t+s(i/n-t)) - \partial_t v_i(t)\|_1 = O(|i/n-t|). \qquad (7.62)$$

The results (7.60), (7.61) and (7.62) imply

$$\begin{aligned}
&\mathbb{E}\big[(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t)\otimes \nabla_\theta^2 \ell(\tilde{Z}_i(i/n),\theta(i/n)) r_n(t)\big] \\
&= \begin{pmatrix} b_n^2\frac{\mu_{K,2}}{2}V(t)\theta''(t) \\ b_n^3\mu_{K,4}\cdot\big\{\frac{1}{6}V(t)\theta^{(3)}(t) + \frac{1}{2}\mathbb{E}[\partial_t\nabla_\theta^2\ell(\tilde{Z}_0(t),\theta(t))]\cdot\theta''(t)\big\}\end{pmatrix} + O(n^{-1}+b_n^4),
\end{aligned}$$

which together with (7.58) gives the result. □

LEMMA 7.11 (Lipschitz properties of Π_n). *Let $s \geq 0$. Suppose that Assumption 2.1 holds with $r \geq 1$ or Assumption 7.16 holds with $r > 1$. Define*

$$\Pi_n(t) := (nb_n)^{-1}\sum_{i=1}^n (M_i^{(2)}(t,i/n) - \mathbb{E}M_i^{(2)}(t,i/n)),$$

where

$$M_i^{(2)}(t,u) = \hat{K}_{b_n}(u-t)\cdot\int_0^1 M_i(t,u)ds\cdot d_u(t),$$

$M_i(u,t) = \nabla_\theta^2\ell(\tilde{Z}_i(u),\theta(t)+sd_u(t))$ and $d_u(t) = \theta(u)-\theta(t)-(u-t)\theta'(t)$. Then there exist come constants $\tilde{C},\iota' > 0$ such that

$$\Big\|\sup_{t\neq t', |t-t'|<\iota'}\frac{|\Pi_n(t)-\Pi_n(t')|}{|t-t'|_1}\Big\|_1 \leq \tilde{C}.$$

PROOF OF LEMMA 7.11. We have

$$\begin{aligned}
&|M_i^{(2)}(t,u) - M_i^{(2)}(t',u)| \\
\leq\ & |\hat{K}_{b_n}(u-t) - \hat{K}_{b_n}(u-t')|\cdot|M_i(t,u)|\cdot|d_u(t)| \\
&+|\hat{K}_{b_n}(u-t')|\cdot|M_i(t,u)-M_i(t',u)|\cdot|d_u(t)| \\
&+|\hat{K}_{b_n}(u-t')|\cdot|M_i(t',u)|\cdot|d_u(t)-d_u(t')|.
\end{aligned}$$

If Assumption 2.1 holds, we have

$$\begin{aligned}
|M_i(t,u)| &\leq \sup_{\theta\in\Theta}|g(\tilde{Z}_i(u),\theta)|,\\
|M_i(t,u)-M_i(t',u)| &\leq \sup_{\theta\in\Theta}\frac{|g(\tilde{Z}_i(u),\theta)-g(\tilde{Z}_i(u),\theta')|}{|\theta-\theta'|_1}\cdot\{|\theta(t)-\theta(t')|_1+|d_u(t)-d_u(t')|_1\},
\end{aligned}$$

As long as $|t-u|<1$ and $|t-t'|$ is small enough, we obtain $|t'-u|\leq 1$. So in the case that either $|t-u|<1$ or $|t'-u|<1$, Lipschitz continuity of $\theta(\cdot),\theta'(\cdot)$ implies that there exists some constant $\tilde{C}>0$ such that $|d_u(t)-d_u(t')|_1\leq\tilde{C}|t-t'|$, $|\theta(t)-\theta(t')|_1\leq\tilde{C}|t-t'|$, $|d_u(t)|_1\leq\tilde{C}$.

This implies

$$\begin{aligned}
|M_i^{(2)}(t,u)-M_i^{(2)}(t',u)| &\leq \tilde{C}b_n^{-1}L_{\hat{K}}\sup_{\theta\in\Theta}|g(\tilde{Z}_i(u),\theta)|\cdot|t-t'|\\
&\quad+2|\hat{K}|_\infty\tilde{C}^2\cdot\sup_{\theta\in\Theta}\frac{|g(\tilde{Z}_i(u),\theta)-g(\tilde{Z}_i(u),\theta')|}{|\theta-\theta'|_1}|t-t'|\\
&\quad+|\hat{K}|_\infty\tilde{C}\cdot\sup_{\theta\in\Theta}|g(\tilde{Z}_i(u),\theta)|\cdot|t-t'|. \qquad (7.63)
\end{aligned}$$

With Lemma 7.1(i) we obtain the result.

Suppose now that Assumption 7.16 holds. As long as $|t-t'|$ is small enough and n is large enough, $|u-t|\leq b_n$ (or $|u-t'|\leq b_n$) and the twice differentiability of $\theta(\cdot)$ imply that $\sup_{\nu\in[0,1]}|\theta(u)-(\theta(t)+\nu d_u(t))|_1<\iota$, $\sup_{\nu\in[0,1]}|\theta(u)-(\theta(t')+\nu d_u(t'))|_1<\iota$. We then obtain

$$\begin{aligned}
|M_i(t,u)| &\leq \sup_{|\theta-\theta(u)|_1<\iota}|\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta)|,\\
|M_i(t,u)-M_i(t',u)| &\leq \bar{C}\cdot\sup_{\theta\neq\theta',|\theta-\theta(u)|_1<\iota,|\theta'-\theta(u)|_1<\iota}\frac{|\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta)-\tilde{g}_{\theta(u)}(\zeta_i,\tilde{X}_i(u),\theta')|}{|\theta-\theta'|_1}\\
&\quad\times\{|\theta(t)-\theta(t')|_1+|d_u(t)-d_u(t')|_1,
\end{aligned}$$

giving appropriate results for (7.63) and thus the assertion with Lemma 7.3. □

LEMMA 7.12. *Let $U_{i,n}(t):=K_{b_n}(i/n-t)\cdot(1,(i/n-t)b_n^{-1})^\top$. Let Assumption 2.1 or 7.16 hold with some $r=2+\varsigma$, $\varsigma>0$. Then it holds that*

$$\begin{aligned}
&\sup_{t\in(0,1)}\big|\nabla_\eta\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t))-\mathbb{E}\nabla_\eta\hat{L}^\circ_{n,b_n}(t,\eta_{b_n}(t))\\
&\qquad-(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t)\otimes\nabla_\theta\ell(\tilde{Z}_i(i/n),\theta(i/n))\big| = O_{\mathbb{P}}(\beta_n b_n^2).
\end{aligned}$$

PROOF. Note that $\mathbb{E}\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) = 0$ by Assumption 2.1(A1),(A3) or Assumption 7.16(A1'),(A3'). Put

$$\begin{aligned}
&\Pi_n(t) \\
:= \quad &(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t) \otimes \big\{[\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + (i/n-t)\theta'(t)) - \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))] \\
&\qquad -\mathbb{E}[\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(t) + (i/n-t)\theta'(t)) - \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))]\big\}.
\end{aligned}$$

We have to prove that $\sup_{t\in\mathcal{T}_n} \big|\Pi_n(t)\big| = O_{\mathbb{P}}(\delta_n b_n^2)$. Define $M_i(t,u) := \int_0^1 \nabla^2_\theta \ell(\tilde{Z}_i(u), \theta(t) + s(\theta(u)-\theta(t)-(u-t)\theta'(t)))ds$ and $M_i^{(2)}(t,u) = U_{i,n}(t)\otimes\big\{M_i(t,u)\{\theta(u)-\theta(t)-(u-t)\theta'(t)\}\big\}$. By a Taylor expansion of $\nabla_\theta \ell$ w.r.t. θ, we have

$$\Pi_n(t) = (nb_n)^{-1}\sum_{i=1}^n (M_i^{(2)}(t, i/n) - \mathbb{E}M_i^{(2)}(t, i/n)).$$

We now apply a similar technique as in the proof of Lemma 7.8(iii), namely we use a chaining argument similar to (7.50) to prove

$$\mathbb{P}\big(\sup_{t\in(0,1)} |\Pi_n(t)| > Q\beta_n b_n^2\big) \to 0,$$

for some $Q > 0$ large enough. Define the discretization $\mathcal{T}_{n,r} := \{l/r : l = 1,\dots,r\}$ with $r = n^5$. By Lemma 7.11, we have with Markov's inequality for $Q > 0$:

$$\mathbb{P}\big(\sup_{|t-t'|\le r^{-1}} |\Pi_n(t) - \Pi_n(t')| > Q\beta_n b_n^2/2\big) = O\big(\frac{b_n^{-2}r^{-1}}{\beta_n b_n^2}\big),$$

which converges to 0. Choose $\alpha = 1/2$. By Lemma 7.5(iii) or Lemma 7.6(iii) applied with $q = 2+s$ (s small enough), we obtain that $\sup_u \Delta_{2+s}^{\sup_t |M^{(2)}(t,u)|}(k) = O(k^{-(1+\gamma)})$. Thus

$$\begin{aligned}
\tilde{W}_{2+s,\alpha} \quad &:= \quad \sup_{u\in[0,1]}\sup_{t\in[0,1]} \|\sup_{t,\eta}|M_i^{(2)}(t,u)|\|_{2+\varsigma,\alpha} = \sup_{m\ge 0}(m+1)^\alpha \Delta_{2+s}^{\sup_t |M^{(2)}(t,u)|}(m) \\
&= \quad O(b_n^2) \qquad (7.64)
\end{aligned}$$

(the constant being independent of n) and

$$\begin{aligned}
\tilde{W}_{2,\alpha} \quad &:= \quad \sup_{t,u}\|M_i^{(2)}(t,u)\|_{2,\alpha} = \sup_{m\ge 0}(m+1)^\alpha \sup_{u\in[0,1]}\sup_t \Delta_2^{M^{(2)}(t,u)}(m) \\
&= \quad O(b_n^2) \qquad (7.65)
\end{aligned}$$

(the constant being independent of n). We now apply Theorem 6.2 of [61] (the proof therein also works for the uniform functional dependence measure) with $q = 2 + s$, $\alpha = 1/2$ to $(M_i^{(2)}(t, i/n))_{t\in\mathcal{T}_{n,r}}$, where $l = 1 \vee \#(\mathcal{T}_{n,r}) \le 5\log(n)$. For Q large enough, we obtain with some constant $C_{\alpha,s} > 0$:

$$
\begin{aligned}
& \mathbb{P}\big(\sup_{t'\in\mathcal{T}_{n,r}} |\Pi_n(t')| \ge Q\beta_n b_n^2/2\big) \\
\le\ & \frac{C_{\alpha,s} n \cdot l^{1+s/2} \tilde{W}_{2+s,\alpha}^{2+s}}{(Q/2)^{2+s}(\beta_n b_n^2 (nb_n))^{2+s}} + C_{\alpha,s}\exp\Big(-\frac{C_{\alpha,s}(Q/2)^2(\beta_n b_n^2(nb_n))^2}{n\tilde{W}_{2,\alpha}^2}\Big) \\
\lesssim\ & n^{-\varsigma/2} + \exp\big(-\frac{(nb_n)b_n^{-1}\log(n)}{n}\big) \\
\to\ & 0,
\end{aligned}
$$

which finishes the proof. □

7.2. *Proofs and Lemmas for the SCB.* From Lemma 1 in [63], we adopt the following result:

LEMMA 7.13. *Let $F_n(t) = \sum_{i=1}^n \hat{K}_{b_n}(t_i - t)V_i$, where $V_i, i \in \mathbb{Z}$ are i.i.d. $N(0, I_{s\times s})$. $b_n \to 0$ and $nb_n/\log^2(n) \to \infty$. Let $m^* = 1/b_n$. Then*

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{1}{\sigma_{\hat{K},0}\sqrt{nb_n}} \sup_{t\in\mathcal{T}_n} |F_n(t)| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)). \tag{7.66}$$

where $B_{\hat{K}}$ is defined in (3.11).

The following lemma is an analogue of Lemma 2 in [63]. Since we use other Gaussian approximation rates from Theorem 3.3, we shortly state the proof for completeness.

LEMMA 7.14. *Let the assumptions and notations from Theorem 3.3 hold. Define*

$$D_{\tilde{h}}(t) := (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n - t)\tilde{h}_i(i/n).$$

Assume that $\Sigma_{\tilde{h}}(t)$ is Lipschitz-continuous and that its smallest eigenvalue is bounded away from 0 uniformly on $[0, 1]$. Assume that $\log(n)^4\big(b_n n^{(2\gamma+\varsigma\gamma-\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)}\big)^{-1} \to 0$ and $b_n \log(n)^{3/2} \to 0$. Then

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sqrt{nb_n}}{\sigma_{\hat{K},0}} \sup_{t\in\mathcal{T}_n} \Big|\Sigma_{\tilde{h}}^{-1}(t)D_{\tilde{h}}(t)\Big| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)), \tag{7.67}$$

PROOF OF LEMMA 7.14. By Theorem 3.3 and summation-by-parts, there exist i.i.d. $V_i \sim N(0, I_{s\times s})$ such that

$$\sup_{t\in(0,1)} |D_{\tilde{h}}(t) - \Xi(t)| = O_{\mathbb{P}}\Big(\frac{n^{\frac{2\varsigma+2\gamma+\gamma\varsigma}{2\varsigma+8\gamma+4\gamma\varsigma}} \log(n)^{\frac{2\gamma(3+\varsigma)}{\varsigma+4\gamma+2\gamma\varsigma}}}{nb_n}\Big) = O_{\mathbb{P}}\Big(\frac{\log(n)^2\big(b_n n^{\frac{2\gamma+\varsigma\gamma-\varsigma}{\varsigma+4\gamma+2\gamma\varsigma}}\big)^{-1/2}}{(nb_n)^{1/2}\log(n)^{1/2}}\Big), \tag{7.68}$$

where $\Xi(t) = (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)\Sigma_{\tilde{h}}(i/n)V_i$. Here, (7.68) is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ due to

$$\log(n)^4\big(b_n n^{(2\gamma+\varsigma\gamma-\varsigma)/(\varsigma+4\gamma+2\gamma\varsigma)}\big)^{-1} \to 0.$$

Since $\Sigma_{\tilde{h}}(\cdot)$ is Lipschitz continuous by Assumption (b), we can use a standard chaining argument in t (as it was done in Lemma 7.12 for $\Pi_n(t)$) and the fact that $(nb_n)^{-1}\sum_{i=1}^n(\Sigma_{\tilde{h}}(i/n) - \Sigma_{\tilde{h}}(t))\hat{K}_{b_n}(i/n-t)V_i \sim N(0, v_n)$, with $|v_n|_\infty \le C\frac{b_n}{n}$ for some constant $C>0$ to obtain

$$\begin{aligned}
&\sup_{t\in(0,1)} |\Xi(t) - (nb_n)^{-1}\Sigma_{\tilde{h}}(t)\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)V_i| \\
= &\sup_{t\in(0,1)} \big|(nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)(\Sigma_{\tilde{h}}(i/n) - \Sigma_{\tilde{h}}(t))V_i\big| \\
= &\; O_{\mathbb{P}}\Big(\frac{b_n\log(n)}{(nb_n)^{1/2}}\Big) = O_{\mathbb{P}}\Big(\frac{b_n\log(n)^{3/2}}{(nb_n)^{1/2}\log(n)^{1/2}}\Big),
\end{aligned} \tag{7.69}$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ due to $b_n\log(n)^{3/2} \to 0$. So the result follows from Lemma 7.13 in view of (7.68) and (7.69). □

PROOF OF THEOREM 3.4. Let $\tilde{k}_i(t) := \nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$ and $\hat{K}(x) = K(x)$ or $\hat{K}(x) = K(x)x$, respectively. Define

$$\Omega_C(t) := (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)A_C(i/n)^{\mathsf{T}}\tilde{k}_i(i/n)$$

and $D_{\tilde{k}}(t) = (nb_n)^{-1}\sum_{i=1}^n \hat{K}_{b_n}(i/n-t)\tilde{k}_i(i/n)$. Similar to the discussion of $\Pi_n(t)$ in the proof of Lemma 7.12 (note that the rates in (7.64) and (7.65) then change to $O(b_n)$ instead of $O(b_n^2)$), we can show that

$$\sup_{t\in(0,1)} |\Omega_C(t) - A_C(t)^{\mathsf{T}}\cdot D_{\tilde{k}}(t)| = O_{\mathbb{P}}(\beta_n b_n) = O_{\mathbb{P}}\Big(\frac{b_n^{1/2}\log(n)}{(nb_n)^{1/2}\log(n)^{1/2}}\Big), \tag{7.70}$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ since $b_n\log(n)^2 \to 0$.

$\tilde{h}_i(t) := A_C(t)^\top \tilde{k}_i(t)$ is a locally stationary process with long-run variance $\Sigma^2_{\tilde{h}}(t) = \Sigma^2_C(t)$. By the result of Lemma 7.14, we have that

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\sqrt{nb_n}}{\sigma_{\hat{K},0}} \sup_{t\in\mathcal{T}_n} \big|\Sigma_C^{-1}(t)\Omega_C(t)\big| - B_{\hat{K}}(m^*) \le \frac{u}{\sqrt{2\log(m^*)}}\Big) = \exp(-2\exp(-u)). \tag{7.71}$$

(i) By Theorem 3.2(i), we have

$$\begin{aligned}
&\sup_{t\in\mathcal{T}_n} \big|V(t)\{\hat{\theta}_{b_n}(t) - \theta(t)\} - b_n^2 \frac{\mu_{K,2}}{2} V(t)\theta''(t) - D_{\tilde{k}}(t)\big| \\
&= O_{\mathbb{P}}\big(b_n^3 + (nb_n)^{-1} b_n^{-1/2}\log(n)^{3/2} + (nb_n)^{-1/2} b_n \log(n)\big) \\
&= O_{\mathbb{P}}\Big(\frac{(nb_n^7\log(n))^{1/2} + (nb_n^2\log(n)^{-4})^{-1/2} + b_n\log(n)^{3/2}}{(nb_n)^{1/2}\log(n)^{1/2}}\Big),
\end{aligned} \tag{7.72}$$

which is $o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2})$ since $nb_n^7\log(n) \to 0$, $nb_n^2\log(n)^{-4} \to \infty$ and $b_n \log(n)^2 \to 0$. Together with (7.70) and (7.71) (with $\hat{K} = K$), this implies (3.9).

(ii) By Theorem 3.2(ii), we have

$$\begin{aligned}
&\sup_{t\in\mathcal{T}_n} \big|\mu_{K,2} V(t) b_n \{\widehat{\theta'}_{b_n}(t) - \theta'(t)\} - b_n^3 \frac{\mu_{K,4}}{2} V(t)\mathrm{bias}(t) - D_{\tilde{k}}(t)\big| \\
&= O_{\mathbb{P}}\big(b_n^4 + (nb_n)^{-1} b_n^{-1/2}\log(n)^{3/2} + (nb_n)^{-1/2} b_n \log(n)\big) \\
&= o_{\mathbb{P}}((nb_n)^{-1/2}\log(n)^{-1/2}),
\end{aligned}$$

as above. Together with (7.70) and (7.71) (with $\hat{K}(x) = K(x)x$), this implies (3.10). □

7.3. *Proofs of Section 4.*

Proof of Proposition 4.1. (i) Lemma 7.8(i),(iii), Lemma 7.9 and the notation therein applied to $g = \nabla^2_\theta \ell$ imply

$$\begin{aligned}
&\sup_{t\in\mathcal{T}_n} |\hat{\mu}_{K,0,b_n}(t)\hat{V}_{b_n}(t) - \hat{\mu}_{K,0,b_n}(t)V(t)| \\
&\le \sup_{t\in\mathcal{T}_n,\eta\in E_n} |G_n^c(t,\eta) - \hat{G}_n(t,\eta)| + \sup_{t\in\mathcal{T}_n,\eta\in E_n} |\hat{G}_n(t,\eta)| \\
&\quad + \sup_{t\in\mathcal{T}_n,\eta\in E_n} |\mathbb{E}\hat{B}_n(t,\eta) - V^\circ(t,\eta)| + \sup_{t\in\mathcal{T}_n} |V^\circ(t,\hat{\eta}_{b_n}) - \hat{\mu}_{K,0,b_n}(t)V(t)| \\
&= O_{\mathbb{P}}((nb_n)^{-1}) + o_{\mathbb{P}}(\beta_n) + O(b_n) + \sup_{t\in\mathcal{T}_n} |V^\circ(t,\hat{\eta}_{b_n}) - \hat{\mu}_{K,0,b_n}(t)V(t)|.
\end{aligned} \tag{7.73}$$

We obtain similar as in the proof of Theorem 3.2(i) ((7.24) therein) that

$$\sup_{t\in\mathcal{T}_n} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^2).$$

Since $\eta \mapsto V^{\circ}(t,\eta)$ is Lipschitz continuous by Lemma 7.1, the result follows from (7.73) and $b_n \log(n) \to 0$.

(ii) follows similarly due to $\nabla_\theta \ell \cdot \nabla_\theta \ell^{\mathsf{T}} \in \mathcal{H}(2M_y, 2M_y, \chi, \bar{\bar{C}})$ with some $\bar{\bar{C}} > 0$.

□

To prove Theorem 4.2, we adopt the methods used in [63]. Let us first assume that $\theta(\cdot)$ and the stationary approximation $\tilde{Z}_i(t)$ is known. Define $D_i := \nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))$, $Q_i := \sum_{j=-m}^{m} D_{i+j}$ and $\Phi_i := Q_i Q_i^{\mathsf{T}}/(2m+1)$. Recall that τ_n is some bandwidth and $\gamma_n = \tau_n + (m+1)/n$. For $t \in \mathcal{I}_n = [\gamma_n, 1-\gamma_n] \subset (0,1)$, define

$$\hat{\Lambda}(t) := \frac{\sum_{i=1}^{n} K_{\tau_n}(i/n - t)\Phi_i}{\sum_{i=1}^{n} K_{\tau_n}(i/n - t)}.$$

Note that $\hat{\Lambda}(t)$ is always positive semi-definite. We have the following convergence result.

THEOREM 7.15. *Suppose that Assumption 2.1 holds with $r = 4$. Assume that $m = m_n \to \infty$, $m = O(n^{1/3})$, $\tau_n \to 0$ and $n\tau_n \to \infty$. Then with $\rho = 1$,*

(i) For fixed $t \in (0,1)$,

$$\|\hat{\Lambda}(t) - \Lambda(t)\|_2 = O\Big(\sqrt{\frac{m}{n\tau_n}} + m^{-1} + \tau_n^{\rho}\Big).$$

(ii) We have

$$\|\sup_{t\in\mathcal{I}_n} |\hat{\Lambda}(t) - \Lambda(t)|\|_2 = O\Big(\sqrt{\frac{m}{n\tau_n^2}} + m^{-1} + \tau_n^{\rho}\Big).$$

If additionally Assumption 2.2(B1),(B3) is fulfilled with $M' = 2M$ and $\nabla_\theta \ell$ is continuously differentiable with $\partial_{z_j} \nabla_\theta \ell \in \mathcal{H}(M_y - 1, M_x - 1, \chi', \hat{\chi}_j \bar{C})$ for all $j \in \mathbb{N}_0$, then one can choose $\rho = 2$.

PROOF OF THEOREM 7.15. Let $\tilde{D}_i(t) := \nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$. By Lemma 7.5(i) applied to $\nabla_\theta \ell$, it holds that $\sup_{t\in[0,1]} \delta_4^{\tilde{D}(t)}(l) = O(l^{-(1+\gamma)})$. By Lemma 7.1, $\sup_{t\in[0,1]} \|\tilde{D}_0(t)\|_4 < \infty$. It is easily seen by Lemma 7.1 applied to $\nabla_\theta \ell \nabla_\theta \ell^{\mathsf{T}} \in \mathcal{H}(2M_y, 2M_x, \chi)$ that $t \mapsto \Lambda(t)$ is Lipschitz-continuous. Thus $D_i = \tilde{D}_i(i/n)$ has the same properties as L_i in [63]. The proof therefore is completely the same as the proof of Theorem 4 in [63] with a modified bias term ($\rho = 1$) and is omitted.

Under the additional assumption, we have $g = \nabla_\theta \ell \nabla_\theta \ell^{\mathsf{T}} \in \mathcal{H}(2M_y, 2M_x, \chi, \bar{C}')$ and $\partial_{z_j}(\nabla_\theta \ell \nabla_\theta \ell) \in \mathcal{H}(2M_y - 1, 2M_y - 1, (\max\{\chi_i', \chi_i\})_{i\in\mathbb{N}}, \bar{C}'\chi_j)$ with some $\bar{C}' > 0$. Application of Lemma 7.4 to g shows that $\Lambda(t)$ is continuously differentiable with Lipschitz continuous derivative. This shows that in this case, one can choose $\rho = 2$. □

PROOF OF THEOREM 4.2. We follow the steps in the proof of Theorem 5 in [63]. Since $\nabla_\theta^2 \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$, we have

$$\sup_{i=1,\dots,n} \sup_{\theta\in\Theta} |\nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta)|_\infty \le 2 \sup_{\theta\in\Theta} |\theta|_\infty \cdot \sup_{i=1,\dots,n} R_{M_y,M_x}(\tilde{Z}_i(i/n)).$$

Note that $\sup_{0\le t\le 1} \|\tilde{Z}_0(t)\|_{4M} < \infty$. By Lemma 7.1, we have $\sup_{i=1,\dots,n} R_{M_y,M_x}(\tilde{Z}_i(i/n)) = O_{\mathbb{P}}(n^{1/4})$ and thus

$$\sup_{i=1,\dots,n} \sup_{\theta\in\Theta} |\nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \theta)|_\infty = O_{\mathbb{P}}(n^{1/4}). \tag{7.74}$$

Put $D_i^\# = \nabla_\theta \ell(\tilde{Z}_i(i/n), \hat{\theta}_{b_n}(i/n))$ and define $Q_i^\#$, $\Phi_i^\#$ and $\Lambda^\#(t)$ accordingly. Then we have

$$\begin{aligned}
&\|\sup_{t\in\mathcal{I}_n} |\tilde{\Lambda}(t) - \Lambda^\#(t)|\|_1 \\
\le\ & \sup_{t\in\mathcal{I}_n} \Big(\sum_{i=1}^n K_{\tau_n}(i/n - t)\Big)^{-1} \cdot |K|_\infty \\
&\quad\times \sum_{i=(m+1)n+\tau_n}^{n} \sup_{j=-m,\dots,m} \big\{\|\tilde{D}_{i+j} - D_{i+j}^\#\|_2 \|\tilde{D}_{i+j}\|_2 + \|\tilde{D}_{i+j} - D_{i+j}^\#\|_2 \|D_{i+j}^\#\|_2\big\}.
\end{aligned}$$

By the results of Lemma 7.1 applied to $\nabla_\theta \ell \in \mathcal{H}(M_y, M_x, \chi, \bar{C})$, we obtain with some constant $\tilde{C} > 0$ that

$$\sup_{i,j} \|\tilde{D}_{i+j}\|_2 \le \tilde{C}, \qquad \|\tilde{D}_{i+j} - D_{i+j}^\#\|_2 \le \tilde{C}\big(n^{-1} + \sum_{l=i+j}^\infty \chi_l\big),$$

and thus $\|\sup_{t\in\mathcal{I}_n} \|\tilde{\Lambda}(t) - \Lambda^\#(t)|\|_1 = O((n\tau_n)^{-1})$.

Define $\beta_n' := (nb_n)^{-1/2}\log(n) + (nb_n)^{-1} + \beta_n b_n^2 + b_n^2$. Then by (7.74) and the fact that $\hat{\theta}_{b_n}(i/n) - \theta(i/n) = O_{\mathbb{P}}(\beta_n')$ from (7.24), we have

$$\sup_{i/n\in\mathcal{I}_n} |D_i - \tilde{D}_i| \le \sup_{i/n\in\mathcal{I}_1} |\nabla_\theta^2 \ell(\tilde{Z}_i(i/n), \bar{\theta}(i/n))| \cdot |\hat{\theta}_{b_n}(i/n) - \theta(i/n)| = n^{1/4}\beta_n'. \tag{7.75}$$

Note that $Q_i/(2m+1)$ is the Nadaraya-Watson-type smoother of the series D_i with the rectangle kernel and bandwidth $\tilde{b}_n = m/n$. By using (7.21) in this context, we obtain

$$\sup_{i/n\in\mathcal{I}_n} \frac{1}{2m+1} |Q_i| = O_{\mathbb{P}}((n\tilde{b}_n)^{-1/2}\log(n)) = O_{\mathbb{P}}(m^{1/2}\log(n)). \tag{7.76}$$

Comparing Φ_i and $\tilde{\Phi}_i$ we obtain

$$(2m+1)(\Phi_i - \tilde{\Phi}_i) = (Q_i - \tilde{Q}_i)Q_i^{\mathsf{T}} + Q_i(Q_i - \tilde{Q}_i)^{\mathsf{T}} - (Q_i - \tilde{Q}_i)(Q_i - \tilde{Q}_i)^{\mathsf{T}}.$$

By equations (7.75) and (7.76), we have $\sup_{i/n\in\mathcal{I}_1} |\Phi_i - \tilde{\Phi}_i| = O_{\mathbb{P}}(\omega_n)$. This implies

$$\sup_{i/n\in\mathcal{I}_n} |\hat{\Lambda}(i/n) - \tilde{\Lambda}(i/n)| = O_{\mathbb{P}}(\omega_n).$$

The results from Theorem 7.15 now imply the assertion. □

Proof of Proposition 4.4. Similar as in the proof of Theorem 3.2(i) by now using the explicit result of Lemma 7.9(a) applied to $g = \ell$ (both for Assumption 2.1 and 7.16), we obtain

$$\sup_{t\in(0,1)} \sup_{\eta\in E_n} |L^{\circ}_{n,b_n}(t,\eta) - \tilde{L}^{\circ}_{b_n}(t,\eta)| = O_{\mathbb{P}}(\beta_n + (nb_n)^{-1}) + O(b_n),$$

where $\tilde{L}^{\circ}_{b_n}(t,\eta) = \int_{-t/b_n}^{(1-t)/b_n} K(x)L(t,\eta_1+\eta_2 x)dx$. By optimality of $\hat{\eta}_{b_n}(t)$,

$$\begin{aligned} 0 &\le L^{\circ}_{n,b_n}(t,\theta(t)) - L^{\circ}_{n,b_n}(t,\hat{\eta}_{b_n}(t)) \\ &\le \tilde{L}^{\circ}_{b_n}(t,\theta(t)) - \tilde{L}^{\circ}_{b_n}(t,\hat{\eta}_{b_n}(t)) + 2\sup_{\eta\in E_n} |L^{\circ}_{n,b_n}(t,\eta) - \tilde{L}^{\circ}_{b_n}(t,\eta)|. \end{aligned}$$

This implies

$$\begin{aligned} &\min\Big\{ \int_{-1}^{0} K(x)\big\{L(t,\hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t,\theta(t))\big\}dx, \\ &\int_{0}^{1} K(x)\big\{L(t,\hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t,\theta(t))\big\}dx\Big\} \le 2\sup_{\eta\in E_n} |L^{\circ}_{n,b_n}(t,\eta) - \tilde{L}^{\circ}_{b_n}(t,\eta)|. \end{aligned} \tag{7.77}$$

Assume that for some $\iota > 0$, $\limsup_{n\to\infty} \sup_{t\in(0,1)} |\hat{\eta}_{b_n}(t) - (\theta(t)^{\mathsf{T}}, 0)^{\mathsf{T}}| \ge \iota$. Then there exists $t \in (0,1)$ such that either (c1)

$$|\hat{\theta}_{b_n}(t) - \theta(t)| \ge \frac{1}{2}|b_n\widehat{\theta}'_{b_n}(t)|$$

and thus $|\hat{\theta}_{b_n}(t) - \theta(t)| > \iota/3$, or (c2)

$$|\hat{\theta}_{b_n}(t) - \theta(t)| < \frac{1}{2}|b_n\widehat{\theta}'_{b_n}(t)|,$$

and thus $|b_n\widehat{\theta}'_{b_n}(t)| > 2\iota/3$.

In case (c1), we have $|\hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x - \theta(t)| \geq |\hat{\theta}_{b_n}(t) - \theta(t)| - |x||b_n\widehat{\theta}'_{b_n}(t)| \geq \frac{\iota}{6}$ for $x \in [0, \frac{1}{4}]$, thus with some $c_0 > 0$,

$$\int_0^1 K(x)\big\{L(t, \hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t, \theta(t))\big\}dx \geq \int_0^{1/4} K(x)\big\{L(t, \hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t, \theta(t))\big\}dx \geq c_0$$

since $\theta \mapsto L(t, \theta)$ is continuous and attains its unique minimum at $\theta = \theta(t)$.

In case (c2), we have $|\hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x - \theta(t)| \geq |x||b_n\widehat{\theta}'_{b_n}(t)| - |\hat{\theta}_{b_n}(t) - \theta(t)| \geq \frac{\iota}{6}$ for $x \in [\frac{3}{4}, 1]$, thus with some $c_0 > 0$,

$$\int_0^1 K(x)\big\{L(t, \hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t, \theta(t))\big\}dx \geq \int_{3/4}^1 K(x)\big\{L(t, \hat{\theta}_{b_n}(t) + b_n\widehat{\theta}'_{b_n}(t)x) - L(t, \theta(t))\big\}dx \geq c_0.$$

In both cases, (7.77) becomes a contradiction. Therefore,

$$\sup_{t \in (0,1)} |\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)| = o_{\mathbb{P}}(1).$$

Using summation-by-parts and Gaussian approximation similar to that presented in Theorem 3.3 for the process $\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n))$, there exists i.i.d. $V_1, V_2, \ldots \sim N(0, I_{s\times s})$ on a richer probability space such that, for π_n as in (3.8)

(7.78)
$$\sup_{t \in (0,1)} \big|(nb_n)^{-1} \sum_{i=1}^n K_{b_n}(i/n - t)(\nabla_\theta \ell(\tilde{Z}_i(i/n), \theta(i/n)) - V_i)\big\}| = O_{\mathbb{P}}((nb_n)^{-1}\pi_n) = O_{\mathbb{P}}((nb_n)^{-1/2}\log(n)).$$

Thus one can replace $\sup_{t \in \mathcal{T}_n}$ by $\sup_{t \in (0,1)}$ in (7.21). A careful examination of the rest of the proof of Theorem 3.2(i) (with Lemma 7.10(7.53) replaced by Lemma 7.10(7.54)) now yields the result

$$\sup_{t \in (0,1)} |\tilde{V}^\circ_{b_n}(t) \cdot (\hat{\eta}_{b_n}(t) - \eta_{b_n}(t)) - \nabla_\eta L^{\circ,c}_{n,b_n}(t, \eta_{b_n}(t))\big| = O_{\mathbb{P}}(\tau_n^{(1)}), \tag{7.79}$$

where (we shortly write $\hat{\mu}_{K,j}(t) = \hat{\mu}_{K,j,b_n}(t)$)

$$\tilde{V}^\circ_{b_n}(t) = \begin{pmatrix} \hat{\mu}_{K,0}(t) & \hat{\mu}_{K,1}(t) \\ \hat{\mu}_{K,1}(t) & \hat{\mu}_{K,2}(t) \end{pmatrix} \otimes V(t).$$

By Lemma 7.8(i), Lemma 7.10 and Lemma 7.12, we obtain furthermore with $U_{i,n}(t) = (K_{b_n}(i/n-t), K_{b_n}(i/n-t)\cdot(i/n-t)b_n^{-1})^{\mathsf{T}}$:

$$\sup_{t\in(0,1)}\Big|\nabla_\eta L_{n,b_n}^{\circ,c}(t,\eta_{b_n}(t)) - b_n^2\begin{pmatrix}\hat\mu_{K,2}(t)\\ \hat\mu_{K,3}(t)\end{pmatrix}\otimes[V(t)\theta''(t)]$$

$$-(nb_n)^{-1}\sum_{i=1}^n U_{i,n}(t)\otimes\nabla_\theta\ell(\tilde Z_i(i/n),\theta(i/n))\Big| = O_{\mathbb{P}}(\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}). \tag{7.80}$$

Recalling the proof of Lemma 7.14, (7.68) and (7.69) and the proof of Theorem 3.4, (7.70) we see that there exist i.i.d. $V_i\sim N(0,I_{s\times s})$ such that both for $\hat K = K$ and $\hat K(x) = K(x)\cdot x$,

$$\sup_{t\in(0,1)}\Big|A_C(t)^{\mathsf{T}}(nb_n)^{-1}\sum_{i=1}^n\hat K_{b_n}(i/n-t)\nabla_\theta\ell(\tilde Z_i(i/n),\theta(i/n)) - \Sigma_C(t)(nb_n)^{-1}\sum_{i=1}^n\hat K_{b_n}(i/n-t)V_i\Big|$$

$$= O_{\mathbb{P}}\Big(\frac{\log(n)^2\big(b_n n^{\frac{2\gamma+\varsigma\gamma-\varsigma}{\varsigma+4\gamma+2\gamma\varsigma}}\big)^{-1/2}}{(nb_n)^{1/2}\log(n)^{1/2}} + \frac{b_n\log(n)^{3/2}}{(nb_n)^{1/2}\log(n)^{1/2}} + \frac{b_n^{1/2}\log(n)}{(nb_n)^{1/2}\log(n)^{1/2}}\Big) =: O_{\mathbb{P}}(w_n). \tag{7.81}$$

With (7.79) and

$$\begin{aligned}\tilde V_{b_n}^{\circ}(t)^{-1} &= \begin{pmatrix}\hat\mu_{K,0}(t) & \hat\mu_{K,1}(t)\\ \hat\mu_{K,1}(t) & \hat\mu_{K,2}(t)\end{pmatrix}^{-1}\otimes V(t)^{-1}\\ &= \frac{1}{\hat\mu_{K,2}(t)N_{b_n}^{(0)}(t)}\begin{pmatrix}\hat\mu_{K,2}(t)V(t)^{-1} & -\hat\mu_{K,1}(t)V(t)^{-1}\\ -\hat\mu_{K,1}(t)V(t)^{-1} & \hat\mu_{K,0}(t)V(t)^{-1}\end{pmatrix},\end{aligned}$$

we obtain:

$$\sup_{t\in(0,1)}\Big|N_{b_n}^{(0)}(t)\cdot\{\hat\theta_{b_n,C}(t)-\theta_C(t)\}$$

$$-\Big[A_C(t)^{\mathsf{T}}\nabla_{\eta_1}L_{n,b_n}^{\circ,c}(t,\eta_{b_n}(t)) - \frac{\hat\mu_{K,1}(t)}{\hat\mu_{K,2}(t)}A_C(t)^{\mathsf{T}}\nabla_{\eta_2}L_{n,b_n}^{\circ,c}(t,\eta_{b_n}(t))\Big]\Big| = O_{\mathbb{P}}(\tau_n^{(1)}).$$

With (7.80) and (7.81), we have

$$\begin{aligned}&\sup_{t\in(0,1)}\Big|N_{b_n}^{(0)}(t)\cdot\{\hat\theta_{b_n,C}(t)-\theta_C(t)\}\\ &\qquad + b_n^2 N_{b_n}^{(1)}(t)\theta_C''(t) - \Sigma_C(t)\big\{Q_{b_n}^{(0)}(t) - \frac{\hat\mu_{K,1}(t)}{\hat\mu_{K,2}(t)}Q_{b_n}^{(1)}(t)\big\}\Big|\\ &= O_{\mathbb{P}}(\tau_n^{(1)} + (\beta_n b_n^2 + b_n^3 + (nb_n)^{-1}) + w_n),\end{aligned}$$

which finishes the proof. $\square$

7.4. *Proofs of the examples in Section 5.*

7.4.1. *Recursively defined models.*

PROOF OF PROPOSITION 5.1. Choose $0 < \tilde{a} < a$ small enough such that (5.3) holds with $\|\zeta_0\|_{(2+\tilde{a})M}$ replaced by $\|\zeta_0\|_{2M}$ (this is possible due to continuity of the term in $\tilde{a} = 0$). Let $q = (2+\tilde{a})M$. Let $\nu = (\nu_0, \dots, \nu_l)^\top$ and $m = (m_1, \dots, m_k)^\top$. As known from Proposition 4.3 and Lemma 4.4 in [14] the process $(Y_i)_{i=1,\dots,n}$ described by (5.1) exists and fulfills Assumption 2.1(A5), (A7) with $\delta_q^Y(i) = O(c^i)$ for some $0 < c < 1$ and $q \geq 1$ if the recursion function $G_\zeta(y,t) := \mu(y,\theta(t)) + \sigma(y,\theta(t))\zeta$ obeys

$$\Big\| \sup_{t\in[0,1]} \sup_{y\neq y'} \frac{|G_{\zeta_0}(y,t) - G_{\zeta_0}(y',t)|}{|y-y'|_{\chi,1}} \Big\|_q \leq 1 \tag{7.82}$$

and

$$\sup_{t\in[0,1]} \|C(\tilde{X}_t(t))\|_q < \infty, \quad C(y) := \sup_{t\neq t'} \frac{\|G_{\zeta_0}(y,t) - G_{\zeta_0}(y,t')\|_q}{|t-t'|}, \tag{7.83}$$

where $|z|_{\chi,1} := \sum_{i=1}^p |z_i|\chi_i$ for some $\chi = (\chi_i)_{i=1,\dots,p} \in \mathbb{R}^p_{\geq 0}$ with $|\chi|_1 = \sum_{i=1}^p \chi_i < 1$. Here, we can bound

$$|\mu(y,\theta) - \mu(y',\theta)| \leq \sum_{i=1}^k |\alpha_i||y-y'|_{\kappa_{i\cdot},1} \leq |y-y'|_{\kappa^{(\mu)}(\alpha),1}, \tag{7.84}$$

where $\chi^{(\mu)}(\alpha) := \sum_{i=1}^k |\alpha_i|\kappa_{i\cdot}$. Furthermore,

$$\begin{aligned}
|\sigma(y,\theta)^2 - \sigma(y',\theta)^2| &\leq \sum_{i=0}^l \beta_i|\nu_i(y) - \nu_i(y')| \\
&\leq \sum_{i=0}^l \sqrt{\beta_i}|y-y'|_{\rho_{i\cdot},1} \cdot \big(\sqrt{\beta_i\nu_i(y)} + \sqrt{\beta_i\nu_i(y')}\big) \\
&\leq \sum_{i=0}^l \sqrt{\beta_i}|y-y'|_{\rho_{i\cdot},1} \cdot \big(\sigma(y,\theta) + \sigma(y',\theta)\big),
\end{aligned}$$

i.e.

$$|\sigma(y,\theta) - \sigma(y',\theta)| \leq |y-y'|_{\chi^{(\sigma)}(\beta),1}, \tag{7.85}$$

where $\chi^{(\sigma)}(\beta) := \sum_{i=1}^l \sqrt{\beta_i}\rho_{i\cdot}$. Define

$$\chi_j^{(\mu,max)} := \sup_t |\chi^{(\mu)}(\alpha(t))_j|, \qquad \chi_j^{(\sigma,max)} := \sup_t |\chi^{(\sigma)}(\beta(t))_j|.$$

Since $\theta(t) = (\alpha(t)^\mathsf{T}, \beta(t)^\mathsf{T})^\mathsf{T} \in \Theta$, we have that

$$\sum_{j=1}^{p} (\chi_j^{(\mu,max)} + \|\zeta_0\|_q \chi_j^{(\sigma,max)}) = \sum_{j=1}^{p} \big(\sup_t |\chi^{(\mu)}(\alpha(t))_j| + \|\zeta_0\|_q \sup_t |\chi^{(\sigma)}(\beta(t))_j| \big) < 1.$$

Define $\chi_j := \chi_j^{(\mu,max)} + \|\zeta_0\|_q \chi_j^{(\sigma,max)}$. Then we have for all $t, y \neq y'$:

$$|\mu(y, \theta(t)) - \mu(y', \theta(t))| + \|\zeta_0\|_q |\sigma(y, \theta(t)) - \sigma(y', \theta(t))| \leq |y - y'|_{\chi,1},$$

which implies (7.82). Proposition 4.3 from [14] now implies the existence of Y_i, the stationary approximation $\tilde{Y}_i(t)$ and $\sup_t \|\tilde{Y}_0(t)\|_q < \infty$. By Lipschitz continuity of θ with constant L_θ, we have

$$|\mu(y, \theta(t)) - \mu(y, \theta(t'))| \leq L_\theta |t - t'| \sum_{i=1}^{k} |m_i(y)|, \tag{7.86}$$

and

$$\begin{aligned} |\sigma(y, \theta(t))^2 - \sigma(y, \theta(t'))^2| &\leq L_\theta |t - t'| \sum_{i=0}^{l} \sqrt{\nu_i(y)} \frac{1}{2\beta_{min}^{1/2}} \big(\sqrt{\beta_i(t)\nu_i(y)} + \sqrt{\beta_i(t')\nu_i(y)} \big) \\ &\leq \frac{L_\theta}{2\beta_{min}^{1/2}} |t - t'| \sum_{i=0}^{l} \sqrt{\nu_i(y)} (\sigma(y, \theta(t)) + \sigma(y, \theta(t'))), \end{aligned}$$

which shows that

$$|\sigma(y, \theta(t)) - \sigma(y, \theta(t'))| \leq \frac{L_\theta}{2\beta_{min}^{1/2}} \sum_{i=0}^{l} \sqrt{\nu_i(y)}. \tag{7.87}$$

Note that (5.2) implies

$$m_i(y), \sqrt{\nu_i(y)} \leq C_1 |y|_1 + C_2,$$

with some constants $C_1, C_2 > 0$. By (7.86), (7.87), we have for $t \neq t'$

$$\begin{aligned} \|G_{\zeta_0}(y, t) - G_{\zeta_0}(y, t')\|_q &\leq |\mu(y, \theta(t)) - \mu(y, \theta(t'))| + \|\zeta_0\|_q |\sigma(y, \theta(t)) - \sigma(y, \theta(t'))| \\ &\leq C_3 |t - t'| \big(1 + |y|_1\big), \end{aligned}$$

with some constant $C_3 > 0$. Since $\sup_t \|\tilde{Y}_0(t)\|_q < \infty$, (7.83) follows.

We now inspect the properties of the function ℓ. First note that the recursion of the stationary approximation,

$$\tilde{Y}_i(t) = \mu(\tilde{X}_i(t), \theta(t)) + \sigma(\tilde{X}_i(t), \theta(t))\zeta_i,$$

implies $\mathbb{E}\tilde{Y}_0(t) = 0$ and $\mathbb{E}\tilde{Y}_0(t)^2 = \mathbb{E}\mu(\tilde{X}_0(t), \theta(t))^2 + \mathbb{E}\sigma(\tilde{X}_0(t), \theta(t))^2 \geq \beta_{min}\nu_{min} > 0$. Furthermore, for $L(t, \theta) := \mathbb{E}\ell(\tilde{Z}_0(t), \theta)$ it holds that

$$\begin{aligned} L(t,\theta) - L(t,\theta(t)) &= \mathbb{E}\Big(\frac{\mu(\tilde{X}_0(t),\theta) - \mu(\tilde{X}_0(t),\theta(t))}{\sigma(\tilde{X}_0(t),\theta)}\Big)^2 \\ &\qquad + \mathbb{E}\Big[\frac{\sigma(\tilde{X}_0(t),\theta(t))^2}{\sigma(\tilde{X}_0(t),\theta)^2} - \log\frac{\sigma(\tilde{X}_0(t),\theta(t))^2}{\sigma(\tilde{X}_0(t),\theta)^2} - 1\Big]. \end{aligned} \tag{7.88}$$

In the following we use the notation $|x|_A^2 := x^\mathsf{T} A x$ for a weighted vector norm. Note that

$$\mathbb{E}\Big(\frac{\mu(\tilde{X}_0(t),\theta) - \mu(\tilde{X}_0(t),\theta(t))}{\sigma(\tilde{X}_0(t),\theta)}\Big)^2 \geq c_0|\alpha - \alpha(t)|^2_{M_1(t)}, \tag{7.89}$$

with $c_0 = (\max_{\theta\in\Theta}\max_i \theta_i^2)^{-1}$ and $M_1(t) := \mathbb{E}[\frac{m(\tilde{X}_0(t))m(\tilde{X}_0(t))^\mathsf{T}}{\nVdash\nu(\tilde{X}(t))\nu(\tilde{X}(t))^\mathsf{T}\nVdash}]$. If $M_1(t)$ was not positive definite, this would imply that there exists $v \in \mathbb{R}^k$ such that $v'M(t)v = 0$, which in turn would imply $v'\mu(\tilde{X}_0(t))\mu(\tilde{X}_0(t))v = 0$ a.s. and thus non-positive definiteness of $\mathbb{E}[\mu(\tilde{X}_0(t))\mu(\tilde{X}_0(t))^\mathsf{T}]$ which is a contradiction to the assumption.
By a Taylor expansion of $f(x) = x - \log(x) - 1$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[\frac{\sigma(\tilde{X}_0(t),\theta(t))^2}{\sigma(\tilde{X}_0(t),\theta)^2} - \log\frac{\sigma(\tilde{X}_0(t),\theta(t))^2}{\sigma(\tilde{X}_0(t),\theta)^2} - 1\Big] \\ \geq\ & \frac{1}{2}\mathbb{E}\Big[\frac{(\sigma(\tilde{X}_0(t),\theta)^2 - \sigma(\tilde{X}_0(t),\theta(t))^2)^2}{(\sigma(\tilde{X}_0(t),\theta)^2 - \sigma(\tilde{X}_0(t),\theta(t))^2)^2 + \sigma(\tilde{X}_0(t),\theta)^4}\Big] \\ \geq\ & \frac{c_0}{10}|\beta - \beta(t)|^2_{M_2(t)}, \end{aligned} \tag{7.90}$$

where $M_2(t) = \mathbb{E}[\frac{\nu(\tilde{X}_0(t))\nu(\tilde{X}_0(t))^\mathsf{T}}{\nVdash\nu(\tilde{X}(t))\nu(\tilde{X}(t))^\mathsf{T}\nVdash}]$ is positive definite by assumption (use a similar argumentation as above). By (7.88), (7.89) and (7.90) we conclude that $\theta \mapsto L(t,\theta)$ is uniquely minimized in $\theta = \theta(t)$. This shows 2.1(A3).

Omitting the arguments $z = (y, x)$ and θ, we have

$$\ell = \frac{1}{2}\Big[\frac{(y - \langle\alpha, m\rangle)^2}{\langle\beta,\nu\rangle} + \log\langle\beta,\nu\rangle\Big], \tag{7.91}$$

$$\nabla_\theta \ell = -\frac{\nabla_\theta m}{\sigma}\Big(\frac{y-m}{\sigma}\Big) + \frac{\nabla_\theta(\sigma^2)}{2\sigma^2}\Big[1 - \Big(\frac{y-m}{\sigma}\Big)^2\Big]$$

$$= \begin{pmatrix} -\frac{m}{\sigma}\Big(\frac{y-m}{\sigma}\Big) \\ \frac{\nu}{2\sigma^2}\big[1 - \big(\frac{y-m}{\sigma}\big)^2\big]\end{pmatrix} = \begin{pmatrix} \frac{m}{\langle\beta,\nu\rangle}(y - \langle\alpha,m\rangle) \\ \frac{\nu}{2\langle\beta,\nu\rangle}\big(1 - \frac{(y-\langle\alpha,m\rangle)^2}{\langle\beta,\nu\rangle}\big)\end{pmatrix}, \tag{7.92}$$

$$\nabla_\theta^2 \ell = \frac{\nabla_\theta m \nabla_\theta m^\mathsf{T}}{\sigma^2} + \big(\frac{y-m}{\sigma}\big)\cdot\Big[\frac{\nabla_\theta m\nabla_\theta(\sigma^2)^\mathsf{T} + \nabla_\theta(\sigma^2)\nabla_\theta m^\mathsf{T}}{\sigma^3} - \frac{\nabla_\theta^2 m}{\sigma}\Big]$$
$$+\frac{\nabla_\theta^2(\sigma^2)}{2\sigma^2}\Big[1 - \big(\frac{y-m}{\sigma}\big)^2\Big] + \frac{\nabla_\theta(\sigma^2)\nabla_\theta(\sigma^2)^\mathsf{T}}{2\sigma^4}\Big[2\Big(\frac{y-m}{\sigma}\Big)^2 - 1\Big]$$
$$= \begin{pmatrix} \frac{mm^\mathsf{T}}{\sigma^2} & \frac{y-m}{\sigma^2}\cdot m\nu^\mathsf{T} \\ \frac{y-m}{\sigma^2}\cdot \nu m^\mathsf{T} & \frac{\nu\nu^\mathsf{T}}{2\sigma^4}\big[2\big(\frac{y-m}{\sigma}\big)^2 - 1\big]\end{pmatrix}$$

$$= \begin{pmatrix} \frac{mm^\mathsf{T}}{\langle\beta,\nu\rangle} & \frac{y-\langle\alpha,m\rangle}{\langle\beta,\nu\rangle^2}\cdot m\nu^\mathsf{T} \\ \frac{y-\langle\alpha,m\rangle}{\langle\beta,\nu\rangle^2}\cdot \nu m^\mathsf{T} & \frac{\nu\nu^\mathsf{T}}{2\langle\beta,\nu\rangle^2}\big[2\frac{(y-\langle\alpha,m\rangle)^2}{\langle\beta,\nu\rangle} - 1\big]\end{pmatrix}. \tag{7.93}$$

Since ζ_1 is independent of $\tilde{X}_0(t) \in \mathcal{F}_0$ and $\mathbb{E}\zeta_1 = 0$, $\mathbb{E}\zeta_1^2 = 1$, we conclude that

$$\mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t))|\mathcal{F}_{t-1}] = \mathbb{E}\Big[-\frac{\mu(\tilde{X}_j(t),\theta(t))}{\sigma(\tilde{X}_0(t),\theta(t))}\zeta_0 + \frac{\nu(\tilde{X}_0(t),\theta(t))}{2\sigma(\tilde{X}_j(t),\theta(t))^2}(1-\zeta_0^2)\big|\mathcal{F}_{t-1}\Big] = 0,$$

i.e. $\nabla_\theta \ell(\tilde{Z}_1(t), \theta(t))$ is a martingale difference sequence, showing that $V(t) = \Lambda(t)$. We furthermore have that (we omit the arguments $(\tilde{X}_0(t), \theta(t))$ of μ, σ in the following):

$$V(t) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)) = \begin{pmatrix} \mathbb{E}\big[\frac{mm^\mathsf{T}}{\langle\beta,\nu\rangle}\big] & 0 \\ 0 & \mathbb{E}\big[\frac{\nu\nu^\mathsf{T}}{2\langle\beta,\nu\rangle^2}\big]\end{pmatrix}.$$

With a similar argumentation as above, we conclude that $V(t)$ is positive definite (which then implies by continuity that the smallest eigenvalue of $V(t)$ is bounded away from 0 uniformly in t). By the martingale difference property, $I(t) = \Lambda(t)$. Omitting the arguments $(\tilde{X}_0(t), \theta(t))$,

$$I(t) = \mathbb{E}[\nabla_\theta(\tilde{Z}_j(t), \theta(t))\nabla_\theta(\tilde{Z}_0(t), \theta(t))^\mathsf{T}]$$
$$= \begin{pmatrix} \mathbb{E}\big[\frac{mm^\mathsf{T}}{\sigma^2}\big] & \mathbb{E}[\zeta_0^3]\cdot\mathbb{E}\big[\frac{m\nu^\mathsf{T}}{2\sigma^3}\big] \\ \mathbb{E}[\zeta_0^3]\cdot\mathbb{E}\big[\frac{\nu m^\mathsf{T}}{2\sigma^3}\big] & \frac{\mathbb{E}[\zeta_0^4]-1}{4}\cdot\mathbb{E}\big[\frac{\nu\nu^\mathsf{T}}{\sigma^4}\big]\end{pmatrix}$$
$$= \mathbb{E}\Big[\frac{1}{\sigma^2}\begin{pmatrix} m \\ \frac{\mathbb{E}[\zeta_0^3]}{2\sigma}\nu\end{pmatrix}^\mathsf{T}\begin{pmatrix} m \\ \frac{\mathbb{E}[\zeta_0^3]}{2\sigma}\nu\end{pmatrix}\Big] + \begin{pmatrix} 0 & 0 \\ 0 & \big(\frac{\mathbb{E}[\zeta_0^4]-\mathbb{E}[\zeta_0^3]^2-1}{4}\big)\mathbb{E}\big[\frac{\nu\nu^\mathsf{T}}{\sigma^4}\big]\end{pmatrix},$$

which is positive semidefinite since $\mathbb{E}[\zeta_0^3] = \mathbb{E}[\zeta_0(\zeta_0^2-1)] \le \mathbb{E}[\zeta_0^2]^{1/2}\mathbb{E}[(\zeta_0^2-1)^2]^{1/2} = (\mathbb{E}[\zeta_0^4]-1)^{1/2}$. Positive definiteness follows from the fact that $(v_1,v_2)^\mathsf{T} I(t)(v_1,v_2) = 0$ implies $\nu^\mathsf{T} v_2 = 0$ a.s. from the last summand and $v_1^\mathsf{T} m + \frac{\mathbb{E}[\zeta_0^3]}{2\sigma} v_2^\mathsf{T}\nu = 0$ a.s. from the first summand, i.e. $v_1^\mathsf{T} m = 0$ a.s. which leads to a contradiction to either the positive definiteness of $\mathbb{E}[\nu\nu^\mathsf{T}]$ or $\mathbb{E}[mm^\mathsf{T}]$. So we obtain that Assumption 2.1(A4) is fulfilled.

A careful inspection of (7.91), (7.92) and (7.93) shows that $\ell, \nabla_\theta \ell, \nabla_\theta^2 \ell \in \mathcal{H}(2,3,\tilde\chi,\tilde C)$ with some $\tilde C > 0$ and $\tilde\chi = (1,\dots,1,0,0,\dots)$ consisting of $\max\{k,l\}$ ones followed by zeros, which shows Assumption 2.1(A1). In the special case $\mu(x,\theta)\equiv 0$, it seems as if no direct improvement of the value M is possible. In the special case of $\sigma(x,\theta)^2 \equiv \beta_0$, we have

$$
\begin{aligned}
\ell &= \frac{1}{2}\Big[\frac{(y-\langle\alpha,m\rangle)^2}{\beta_0} + \log\beta_0\Big],\\
\nabla_\theta \ell &= \begin{pmatrix} \frac{m}{\beta_0}(y-\langle\alpha,m\rangle)\\ \frac{1}{2\beta_0}\big(1-\frac{(y-\langle\alpha,m\rangle)^2}{\beta_0}\big)\end{pmatrix},\\
\nabla_\theta^2 \ell &= \begin{pmatrix} \frac{mm^\mathsf{T}}{\beta_0} & \frac{y-\langle\alpha,m\rangle}{\beta_0^2} m\\ \frac{y-\langle\alpha,m\rangle}{\beta_0^2} m^\mathsf{T} & \frac{1}{2\beta_0^2}\big[2\frac{(y-\langle\alpha,m\rangle)^2}{\beta_0}-1\big]\end{pmatrix},
\end{aligned}
$$

which implies that $\ell, \nabla_\theta \ell, \nabla_\theta^2 \ell \in \mathcal{H}(2,2,\tilde\chi,\tilde C)$.

Now suppose that Assumption 2.2(B1) is fulfilled. We use results from Section 4 in [14] to show that the first derivative process $\partial_t \tilde Y_i(t)$ exists and fulfills a Lipschitz condition. By assumption, with some constant $C>0$,

$$
\begin{aligned}
&|\partial_{x_j} G_{\zeta_0}(x,t) - \partial_{x_j} G_{\zeta_0}(x',t)|\\
\le\ & |\langle\alpha(t), \partial_{x_j} m(x) - \partial_{x_j} m(x')\rangle|\\
&\quad + \Big|\frac{\langle\beta(t),\partial_{x_j}\nu(x)\rangle}{2\langle\beta(t),\nu(x)\rangle^{1/2}} - \frac{\langle\beta(t),\partial_{x_j}\nu(x')\rangle}{2\langle\beta(t),\nu(x')\rangle^{1/2}}\Big|\cdot|\zeta_0|\\
\le\ & |\alpha(t)|_\infty C|x-x'|_1\\
&\quad + |\zeta_0|\cdot\Big(\frac{1}{2\beta_{min}^{1/2}}|\langle\beta(t),\partial_{x_j}\nu(x)-\partial_{x_j}\nu(x')\rangle|\\
&+\frac{|\langle\beta(t),\partial_{x_j}\nu(x')\rangle|}{2}\frac{|\langle\beta(t),\nu(x)-\nu(x')\rangle|}{\langle\beta(t),\nu(x)\rangle^{1/2}\langle\beta(t),\nu(x')\rangle^{1/2}(\langle\beta(t),\nu(x)\rangle^{1/2}+\langle\beta(t),\nu(x')\rangle^{1/2})}\Big).
\end{aligned}
$$

By assumption,

$$
|\langle\beta(t),\partial_{x_j}\nu(x)-\partial_{x_j}\nu(x')\rangle| \le C|\beta(t)|_\infty|x-x'|_1.
$$

Furthermore,

$$\begin{aligned}|\langle\beta(t),\nu(x)-\nu(x')\rangle| &\leq \sum_{i=1}^{l}\beta_i(t)|\sqrt{\nu_i(x)}-\sqrt{\nu_i(x')}|\cdot|\sqrt{\nu_i(x)}+\sqrt{\nu_i(x')}| \\ &\leq |\beta(t)|_\infty|x-x'|_1\sum_{i=1}^{l}\big(\sqrt{\nu_i(x)}+\sqrt{\nu_i(x')}\big).\end{aligned}$$

Since each component of $\beta(t)$ is lower bounded by β_{min} and therefore $\langle\beta(t),\nu(x)\rangle^{1/2}\geq\beta_{min}\sqrt{\nu_i(x)}$ for each i, we conclude that

$$\frac{|\langle\beta(t),\nu(x)-\nu(x')\rangle|}{\langle\beta(t),\nu(x)\rangle^{1/2}(\langle\beta(t),\nu(x)\rangle^{1/2}+\langle\beta(t),\nu(x')\rangle^{1/2})}\leq C|x-x'|_1,$$

with some constant $C>0$. Finally, let e_j be the j-th unit vector in $\mathbb{R}^k$. Notice that for all i,

$$\frac{|\partial_{x_j}\nu_i(x)|}{\nu_i(x)^{1/2}}=2|\partial_{x_j}\sqrt{\nu_i(x)}|\leq 2\lim_{h\to 0}\frac{|\sqrt{\nu_i(x)}-\sqrt{\nu_i(x+he_j)}|}{|h|}\leq\lim_{h\to 0}\frac{|he_j|_1}{|h|}\leq 1.$$

This shows that $\frac{|\langle\beta(t),\partial_{x_j}\nu(x')\rangle|}{\langle\beta(t),\nu(x')\rangle^{1/2}}$ is bounded and we obtain that for some constant >0,

$$|\partial_{x_j}G_{\zeta_0}(x,t)-\partial_{x_j}G_{\zeta_0}(x',t)|\leq C(1+|\zeta_0|)|x-x'|_1.$$

With similar but simpler arguments we obtain that

$$|\partial_t G_{\zeta_0}(x,t)-\partial_t G_{\zeta_0}(x',t)|\leq C(1+|\zeta_0|)|x-x'|_1$$

and

$$\frac{|\partial_{x_j}G_{\zeta_0}(x,t)-\partial_{x_j}G_{\zeta_0}(x,t')|}{|t-t'|\cdot|x|_1}\leq C(1+|\zeta_0|),\qquad\frac{|\partial_t G_{\zeta_0}(x,t)-\partial_t G_{\zeta_0}(x,t')|}{|t-t'|\cdot|x|_1}\leq C(1+|\zeta_0|).$$

By Theorem 4.8 and Proposition 4.11 in [14], we obtain 2.2(B3) with $M_2=2M$.
Finally, straightforward calculations show that each component of $\partial_{x_j}\nabla_\theta^2\ell$ and $\nabla_\theta^3\ell$ is in $\mathcal{H}(M_2,M_2,\tilde{\chi},\bar{C})$ with $\tilde{\chi}=(1,\dots,1,0,\dots,0)$ consisting of p ones. This shows Assumption 2.2(B2).

□

7.4.2. *tvGARCH.* To make use of independencies occuring in the analysis, let us introduce the class $\mathcal{H}^{mult}_{s,\iota}(M_y, M_x, \chi, \bar{C})$ for $s \geq 0$ which consists of functions $g : \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \Theta$ such that $\sup_{\theta\in\Theta} \frac{|g(y,0,\theta)|}{1+|y|^{M_y}} \leq \bar{C}$ and

$$\sup_{|\theta-\tilde{\theta}|_1<\iota} \sup_{y} \sup_{x\neq x'} \frac{|g(y,x,\theta)-g(y,x',\theta)|}{|x-x'|_{\chi,1}(R_{M_y-1,M_x-1}(1,x)^{1+s}+R_{M_y-1,M_x-1}(1,x')^{1+s})(1+|y|^{M})} \leq \bar{C},$$

$$\sup_{x,y} \sup_{\theta\neq\theta',|\theta-\tilde{\theta}|_1<\iota,|\theta'-\tilde{\theta}|_1<\iota} \frac{|g(y,x,\theta)-g(y,x,\theta')|}{|\theta-\theta'|_1 R_{M_y,M_x}(1,x)^{1+s}(1+|y|^{M_y})} \leq \bar{C}.$$

Let $|x|_{\chi,s} := (\sum_{j=1}^{\infty} \chi_j |x|^s)^{1/s}$.

ASSUMPTION 7.16 (Heteroscedastic recursively defined time series case). *Let ζ_i, $i \in \mathbb{Z}$ be an i.i.d. sequence. Assume that the stationary approximation $\tilde{Y}_i(t)$ of Y_i evolves according to*

$$\tilde{Y}_i(t) = F(\tilde{X}_i(t), \theta(t), \zeta_i)$$

where $\tilde{X}_i(t) = (\tilde{Y}_{i-1}(t), \tilde{Y}_{i-2}(t), ...)$.
Let

$$\tilde{\ell}_{\tilde{\theta}}(y, x, \theta) := \ell(F(x, \tilde{\theta}, y), x, \theta).$$

Suppose that for some $r \geq 2$ and $\gamma > 1$,

(A1') ℓ is twice continuously differentiable w.r.t. θ. There exists $M \geq 1$ such that for each $s > 0$, there exist $\chi^{(s)} = (\chi^{(s)}_j)_{j=1,2,...}$ with $\chi^{(s)}_j = O(j^{-(1+\gamma)})$ and $\bar{C}^{(s)} > 0$, such that

- $\ell, \nabla_\theta \ell, \nabla^2_\theta \ell \in \mathcal{H}_s(2M, 2M, \chi^{(s)}, \bar{C}^{(s)})$.
- (7.94)
$$\sup_{\theta} \sup_{z\neq z'} \frac{|\ell(z,\theta)-\ell(z',\theta)|}{|z-z'|^s_{\hat{\chi}^{(s)},s}(R_{M,M}(z)+R_{M,M}(z'))+|z-z'|_{\hat{\chi}^{(s)},1}(R_{M-1,M-1}(z)^{1+s}+R_{M-1,M-1}(z')^{1+s})} \leq \bar{C}^{(s)}.$$
- *There exists $\iota > 0$ such that $\nabla_\theta \tilde{\ell}, \nabla^2_\theta \tilde{\ell} \in \mathcal{H}^{mult}_{s,\iota}(M, M, \chi^{(s)}, \bar{C}^{(s)})$*

(A2') (A2) holds,
(A3') (A3) holds,
(A4') (A4) holds,
(A5') (A5) holds with (2.12) and $\|\zeta_0\|_{rM} \leq D$.
(A6') $X^c_i = (Y_{i-1}, Y_{i-2}, ..., Y_1, 0, 0, ...)$

(A7') $\sup_{t\in[0,1]} \delta_{rM}^{\tilde{Y}(t)}(k) = O(\rho^k)$ *with some* $\rho \in (0,1)$.

ASSUMPTION 7.17 (Heteroscedastic recursively defined time series case). *Suppose that there exists* $M' \geq 2$ *such that* $M' \leq rM$ *and for all* $s > 0$ *there exist absolutely summable* $(\chi')^{(s)} = ((\chi')_j^{(s)})_{j=1,2,\ldots}$ *and* $\iota > 0$ *such that*

(B1') *(B1) holds,*

(B2') $\nabla_\theta^2 \ell$ *is continuously differentiable. It holds that* $\nabla_\theta^3 \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M', M', \chi^{(s)}, \bar{C}^{(s)})$, *and for all* $i \in \mathbb{N}_0$, $\partial_{x_l} \nabla_\theta^2 \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M'-1, M'-1, (\chi')^{(s)}, \bar{C}^{(s)}\chi_l^{(s)})$.

(B3') *(B3) holds.*

PROOF OF PROPOSITION 5.2. We abbreviate $M_i(t) := M_i(\theta(t))$. Note that $(t,\tilde{q}) \mapsto \lambda_{max}(\|M_0(t)\|_{\tilde{q}})$ is continuous; therefore there exists $0 < \tilde{a} < a$ such that $q := 2 + \tilde{a}$ fulfills

$$\sup_{t\in[0,1]} \lambda_{max}(\|M_0(t)\|_q) < 1.$$

Let $M = 1$. Fix $t \in [0,1]$. Consider the recursion of the corresponding stationary approximation

$$\begin{aligned} \tilde{Y}_i(t) &= \tilde{\sigma}_i(t)^2 \zeta_i^2, \\ \tilde{\sigma}_i(t)^2 &= \alpha_0(t) + \sum_{j=1}^m \alpha_j(t)\tilde{Y}_{i-j}(t) + \sum_{j=1}^l \beta_j(t)\tilde{\sigma}_{i-j}(t)^2. \end{aligned} \tag{7.95}$$

Define

$$\begin{aligned} \tilde{P}_i(t) &:= (\tilde{Y}_i(t), \ldots, \tilde{Y}_{i-m+1}(t), \tilde{\sigma}_i(t)^2, \ldots, \tilde{\sigma}_{i-l+1}(t)^2)^{\mathsf{T}}, \\ a_i(t) &:= (\alpha_0(t)\zeta_i^2, 0, \ldots, 0, \alpha_0(t), 0, \ldots, 0)^{\mathsf{T}}. \end{aligned}$$

For brevity, let $M_i(t) = M_i(\theta(t))$. Following Section 3.1 in [54], the model (7.95) admits the representation

$$\tilde{P}_i(t) = M_i(t)\tilde{P}_{i-1}(t) + a_i(t). \tag{7.96}$$

Therefore, $\tilde{P}_i(t) = G_{\zeta_i}(\tilde{P}_{i-1}(t), t)$ with $G_{\zeta_i}(y,t) = M_i(t) \cdot y + a_i(t)$. Let $W_n(y,t) := G_{\zeta_n}(G_{\zeta_{n-1}}(\ldots G_{\zeta_1}(y,t)\ldots))$. Then we have

$$W_n(y) - W_n(y') = M_n(t)M_{n-1}(t) \cdot \ldots \cdot M_1(t) \cdot (y - y').$$

For a vector $v \in \mathbb{R}^p$ and $q \geq 2$ it holds that $\||v|_2\|_q \leq |\|v\|_q|_2$. Thus we have (let $|\cdot|_2$ denote the spectral norm of a matrix):

$$\||W_n(y,t) - W_n(y',t)|_2\|_q \leq |\|M_0(t)\|_q^n(y-y')|_2 \leq |\|M_0(t)\|_q^n|_2|y-y'|_2.$$

$\|M_0(t)\|_q$ is diagonalizable over the complex numbers and therefore has a Jordan form. Since $\sup_{t\in[0,1]} \lambda_{max}(\|M_0(t)\|_q) < 1$, we conclude that

$$\sup_{t\in[0,1]} |\|M_0(t)\|_q^n|_2 \leq C \cdot c^n \tag{7.97}$$

with some $0 < c < 1$ and some constant $C > 0$. By Theorem 2 in [55], we obtain existence and a.s. uniqueness of $\tilde{Y}_i(t) = H(t, \mathcal{F}_i)$, $\sup_{t\in[0,1]} \|\tilde{Y}_0(t)\|_q < \infty$ and $\sup_{t\in[0,1]} \delta_q^{\tilde{Y}(t)}(k) = \|\tilde{Y}_i(t) - \tilde{Y}_i(t)^*\|_q = O(c^k)$. This shows Assumption 7.16(A7').

(7.96) implies the explicit representation

$$\tilde{P}_i(t) = \sum_{k=0}^{\infty}\Big(\prod_{j=0}^{k-1} M_{i-j}(t)\Big)a_{i-k}(t). \tag{7.98}$$

We therefore have for $t, t' \in [0,1]$:

$$\begin{aligned}\|\tilde{P}_i(t) - \tilde{P}_i(t')\|_q \quad \leq \quad & \sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\Big(\prod_{0\leq j<l}\|M_{i-j}(t)\|_q\Big)\|M_{i-l}(t) - M_{i-l}(t')\|_q \\ & \qquad \times\Big(\prod_{l<j\leq k-1}\|M_{i-j}(t)\|_q\Big)\cdot\|a_{i-k}(t)\|_q \\ & + \sum_{k=0}^{\infty}\Big(\prod_{j=0}^{k-1}\|M_{i-j}(t)\|_q\Big)\|a_{i-k}(t) - a_{i-k}(t')\|_q.\end{aligned} \tag{7.99}$$

By Lipschitz continuity of $\theta(\cdot)$ and (7.97), we have

$$\|a_0(t) - a_0(t')\|_q = |\alpha_0(t) - \alpha_0(t')|(\|\zeta_0^2\|_q, 0, \ldots, 0, 1, 0, \ldots, 0)^\mathsf{T} = O(|t-t'|)$$

and

$$\|M_0(t) - M_0(t')\|_q = (\|\zeta_0^2\|_q|f(\theta(t)) - f(\theta(t'))|, 0, \ldots, 0, |f(\theta(t)) - f(\theta(t'))|, 0, \ldots, 0)^\mathsf{T} = O(|t-t'|).$$

We conclude from the first component of (7.99), that for all $t, t' \in [0,1]$:

$$\|\tilde{Y}_i(t) - \tilde{Y}_i(t')\|_q \leq C \cdot |t-t'|, \tag{7.100}$$

with some constant $C > 0$.

Put $P_i = (Y_i, \ldots, Y_{i-m+1}, \sigma_i^2, \ldots, \sigma_{i-l+1}^2)^{\mathsf{T}}$. Similarly to (7.96), we have

$$P_i = M_i(i/n)P_{i-1} + a_i(i/n), \quad i = 1, \ldots, n. \tag{7.101}$$

Note that i iterations of (7.101) lead to $P_0 = \tilde{P}_0(0)$, thus existence of Y_i follows from existence of $\tilde{Y}_i(0)$. We have

$$\begin{aligned} \|P_i - \tilde{P}_i(i/n)\|_q &\le \|M_i(i/n)\|_q \|P_{i-1} - \tilde{P}_{i-1}(i/n)\|_q \\ &\le \|M_0(i/n)\|_q \|P_{i-1} - \tilde{P}_{i-1}((i-1)/n)\|_q \\ &\qquad + \|M_0(i/n)\|_q \|\tilde{P}_0(i/n) - \tilde{P}_0((i-1)/n)\|_q. \end{aligned}$$

Iteration of this inequality leads to

$$\|P_i - \tilde{P}_i(i/n)\|_q \le \sum_{k=1}^{i} \Big(\prod_{j=0}^{k} \|M_0((i-j)/n)\|_q \Big) \|\tilde{P}_0((i-k)/n) - \tilde{P}_0((i-k-1)/n)\|_q.$$

Due to (7.97) and (7.100), we conclude from the first component that $\|Y_i - \tilde{Y}_i(i/n)\|_q = O(n^{-1})$. This shows Assumption 7.16(A5').

Let $\Sigma(x,\theta) := (\sigma(x,\theta)^2, \ldots, \sigma(x_{(l-1)\to}, \theta)^2)^{\mathsf{T}}$ and $A(x,\theta) := (\alpha_0 + \sum_{j=1}^m \alpha_j x_j, \ldots, \alpha_0 + \sum_{j=1}^m \alpha_j x_{j+l-1})^{\mathsf{T}}$, and

$$B(\theta) = \begin{pmatrix} \beta_1 & \ldots & \ldots & \ldots & \beta_l \\ 1 & 0 & \ldots & \ldots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

As said in Theorem 2.1 in [38], $\lambda_{max}(\mathbb{E}M_0(\theta)^{\otimes 2}) < 1$ is a necessary and sufficient condition for the corresponding GARCH process with parameters θ to have 4-th moments. We conclude that $\lambda_{max}(\mathbb{E}M_0(\theta)) < 1$ which by Proposition 1 in [23] implies $\lambda_{max}(B(\theta)) < 1$. We have the explicit representation

$$\sigma(x,\theta)^2 = \sum_{k=0}^{\infty} \big(B(\theta)^k A(x_{k\to}, \theta)\big)_1. \tag{7.102}$$

Since $A(0,\theta) = (\alpha_0, 0, ..., 0)^{\mathsf{T}}$, we have

$$\sigma(0,\theta)^2 = \alpha_0 \sum_{k=0}^{\infty} (B(\theta)^k)_{11}.$$

From (7.102) we also obtain that

$$\sigma(x,\theta)^2 = c_0(\theta) + \sum_{j=1}^{\infty} c_j(\theta)\cdot x_j, \tag{7.103}$$

where $c_j(\theta) \geq 0$ satisfies

$$\sup_{\theta\in\Theta} |c_j(\theta)| \leq C\cdot \rho^j \tag{7.104}$$

with some $\rho \in (0,1)$ and $c_0(\theta) \geq \sigma_{min}^2 > 0$ (due to $\alpha_0 \geq \alpha_{min} > 0$). Due to the explicit representation (7.102) with geometrically decaying summands, it is easy to see that $\sigma(x,\theta)^2$ is four times continuously differentiable w.r.t. θ with

$$\nabla_\theta^k(\sigma(x,\theta)^2) = \nabla_\theta^k c_0(\theta) + \sum_{j=1}^{\infty} \nabla_\theta^k c_j(\theta)\cdot x_j, \qquad k \in \{0,1,2,3,4\}, \tag{7.105}$$

where $(\nabla_\theta^k c_j(\theta))_j$ is still geometrically decaying with $\sup_{\theta\in\Theta} |\nabla_\theta^k c_j(\theta)|_\infty \leq C\cdot\rho^j$, say.

From (7.105) we conclude that (component-wise) for $k = 0,1,2,3$:

$$|\nabla_\theta^k(\sigma(x,\theta)^2) - \nabla_\theta^k(\sigma(x',\theta)^2)| \leq C|x-x'|_{(\rho^j)_j,1}, \tag{7.106}$$

$$|\nabla_\theta^k(\sigma(x,\theta)^2) - \nabla_\theta^k(\sigma(x,\theta')^2)| \leq |\theta-\theta'|_1\cdot\sup_{\theta\in\Theta}|\nabla_\theta^{k+1}(\sigma(x,\theta)^2)|_\infty \leq C|\theta-\theta'|_1\cdot|x|_{(\rho^j)_j,1}. \tag{7.107}$$

We obtain that $\ell(y,x,\theta)$ is four times continuously differentiable and

$$\begin{aligned}
\ell(y,x,\theta) &= \frac{1}{2}\Big(\frac{y}{\sigma(x,\theta)^2} + \log(\sigma(x,\theta)^2)\Big),\\
\nabla_\theta \ell(y,x,\theta) &= \frac{\nabla_\theta(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2}\Big(1 - \frac{y}{\sigma(x,\theta)^2}\Big),\\
\nabla_\theta^2 \ell(y,x,\theta) &= \Big[-\frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^{\mathsf{T}}}{2\sigma(x,\theta)^4} + \frac{\nabla_\theta^2(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2}\Big]\Big(1-\frac{y}{\sigma(x,\theta)^2}\Big)\\
&\qquad + \frac{\nabla_\theta(\sigma(x,\theta)^2)\nabla_\theta(\sigma(x,\theta)^2)^{\mathsf{T}}}{2\sigma(x,\theta)^4}\cdot\frac{y}{\sigma(x,\theta)^2}.
\end{aligned}$$

It was shown in the proof of Theorem 2.1 in [23], that $\theta \mapsto L(t,\theta) = \mathbb{E}\ell(\tilde{Z}_0(t),\theta)$ is uniquely minimized in $\theta = \theta(t)$, which shows Assumption 7.16(A3'). As in the proof of Example 5.1, we obtain that

$$V(t) = \mathbb{E}\Big[\frac{\nabla_\theta(\sigma(\tilde{X}_0(t),\theta(t))^2)\nabla_\theta(\sigma(\tilde{X}_0(t),\theta(t))^2)^{\mathsf{T}}}{2\sigma(\tilde{X}_0(t),\theta(t))^4}\Big] = I(t)\frac{2}{\mathbb{E}\zeta_0^4 - 1}.$$

Furthermore,

$$\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t)) = \frac{\nabla_\theta(\sigma(\tilde{X}_i(t), \theta(t))^2)}{2\sigma(\tilde{X}_i(t), \theta(t))^2}\{1 - \zeta_i^2\},$$

which shows that $\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t))$ is a martingale difference sequence w.r.t. $\mathcal{F}_i$. Thus $\Lambda(t) = I(t)$. It was shown in the proof of Theorem 2.2 in [23] that $V(t)$ is positive definite for each $t \in [0,1]$. By continuity, we conclude that Assumption 7.16(A4') is fulfilled.

Proof of Assumption 7.16(A1'): It holds that

$$2|\ell(y,x,\theta)-\ell(y',x',\theta)| \le |y-y'|\cdot\frac{1}{\sigma(x,\theta)^2}+|y'|\cdot\Big|\frac{1}{\sigma(x,\theta)^2}-\frac{1}{\sigma(x',\theta)^2}\Big|+|\log(\sigma(x,\theta)^2)-\log(\sigma(x',\theta)^2)|.$$

Since $\sigma(x,\theta)^2 \ge \sigma_{min}^2 > 0$, Lipschitz continuity of log on $[\sigma_{min}, \infty)$ and (7.104), there exists some constant $C' > 0$ such that

$$2|\ell(y,x,\theta) - \ell(y',x',\theta)| \le C'(|y-y'| + |x-x'|_{(\rho^j)_j,1}) + |y'| \cdot \Big|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\Big|. \tag{7.108}$$

Note that

$$\Big|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\Big| \le \frac{\sum_{j=0}^\infty c_j(\theta)|x_j.x_j'|}{\sigma(x,\theta)^2\sigma(x',\theta)^2} \le \sum_{j=1}^\infty \frac{c_j(\theta)|x_j - x_j'|}{(\sigma_{min}^2 + c_j(\theta)x_j)(\sigma_{min}^2 + c_j(\theta)x_j')}$$
$$\le \frac{1}{\sigma_{min}^2}\sum_{j=1}^\infty \frac{c_j(\theta)|x_j - x_j'|}{\sigma_{min}^2 + c_j(\theta)|x_j - x_j'|}.$$

The last step holds due to the following argument: It holds either $|x_j - x_j'| \le x_j$ or $|x_j - x_j'| \le x_j'$ since $x_j, x_j' \ge 0$. Therefore, one factor in the denominator can be lower bounded by σ_{min} and the other one by $\sigma_{min} + c_j(\theta)|x_j - x_j'|$. Following the ideas of [23], for arbitrarily small $s > 0$ we use the inequality $\frac{x}{1+x} \le x^s$ to obtain

$$\Big|\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)^2}\Big| \le \frac{1}{\sigma_{min}^{4+2s}}\sum_{j=1}^\infty c_j(\theta)^s|x_j - x_j'|^s \le \frac{C}{\sigma_{min}^{4+2s}}|x-x|_{(\rho^{js})_j,s}$$

Together with (7.108), we obtain (7.94).

Using directly (7.108) and (7.106), we obtain

$$\sup_{\theta\in\Theta}\sup_{z\neq z'}\frac{|\ell(z,\theta) - \ell(z',\theta)|}{|z-z'|_{(\rho^j)_j,1}\cdot(1 + R_{2M-1,2M-1}(z) + R_{2M-1,2M-1}(z'))} < \infty.$$

Note that with some constant $C' > 0$,

$$\begin{aligned} 2|\ell(z,\theta) - \ell(z,\theta')| &\le |y| \cdot \Big[\frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x,\theta')^2}\Big] + |\log(\sigma(x,\theta)^2) - \log(\sigma(x,\theta')^2)| \\ &\le C'(1 + |y|) \cdot |\sigma(x,\theta)^2 - \sigma(x,\theta')^2|. \end{aligned}$$

Together with (7.107), we obtain

$$\sup_{\theta \neq \theta'} \sup_{z} \frac{|\ell(z,\theta) - \ell(z,\theta')|}{|\theta - \theta'|_1 \cdot (1 + R_{2M,2M}(z) + R_{2M,2M}(z'))} < \infty.$$

This shows $\ell \in \mathcal{H}_0(2M, 2M, (\rho^j)_j, \bar{C})$ with some suitably chosen $\bar{C} > 0$.

Let $s > 0$ be arbitrary. It was shown in [23], (4.25) therein that with some small $\iota > 0$ only depending on s, Θ, it holds that

$$\sup_{|\tilde{\theta} - \theta| < \iota} \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2} \leq \bar{C}(1 + |x|^s_{(\rho^{js})_j, s}). \tag{7.109}$$

Similarly, one can obtain for $k = 1, 2, 3$ that

$$\sup_{|\tilde{\theta} - \theta| < \iota} \frac{\nabla_\theta^k \sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2} \leq \bar{C}(1 + |x|^s_{(\rho^{js})_j, s}). \tag{7.110}$$

In the following we show that $\nabla_\theta \ell \in \mathcal{H}_s(2M, 2M, (\rho^j)_j, \bar{C})$ with some suitably chosen $\bar{C} > 0$. We have (component-wise):

$$\begin{aligned} & 2|\nabla_\theta \ell(y,x,\theta) - \nabla_\theta \ell(y',x',\theta)| \\ \leq \;& |y - y'| \cdot \frac{1}{\sigma_{min}^2} \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} + |y'| \cdot \Big| \frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x',\theta)} \Big| \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} \\ & + \Big(1 + \frac{|y'|}{\sigma_{min}^2}\Big) \cdot \Big(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x',\theta)^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x',\theta)^2)|}{\sigma(x',\theta)^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \Big). \end{aligned}$$

Using (7.106) and (7.110), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$2|\nabla_\theta \ell(y,x,\theta) - \nabla_\theta \ell(y',x',\theta)| \leq \bar{C}|z - z'|_{(\rho^j)_j, 1} \cdot (1 + R_{2M-1,2M-1}(z) + R_{2M-1,2M-1}(z'))^{1+s}. \tag{7.111}$$

We have (component-wise):

$$\begin{aligned} & 2|\nabla_\theta \ell(z,\theta) - \nabla_\theta \ell(z,\theta')| \\ \leq \;& |y| \cdot \Big| \frac{1}{\sigma(x,\theta)^2} - \frac{1}{\sigma(x,\theta')} \Big| \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2} \\ & + \Big(1 + \frac{|y|}{\sigma_{min}^2}\Big) \cdot \Big(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x,\theta')^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x,\theta')^2)|}{\sigma(x,\theta')^2 \sigma_{min}^2} |\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \Big). \end{aligned}$$

Using (7.106) and (7.110), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$2|\nabla_\theta \ell(z,\theta) - \nabla_\theta \ell(z,\theta')| \leq \bar{C}|\theta - \theta'|_1 \cdot (1 + R_{2M,2M}(z) + R_{2M,2M}(z'))^{1+s}. \tag{7.112}$$

We conclude from (7.111) and (7.112) that $\nabla_\theta \ell \in \mathcal{H}_s(2M, 2M, (\rho^j)_j, \bar{C})$. The proof for $\nabla_\theta^2 \ell$ is similar in view of (7.106), (7.107) and (7.110) and therefore omitted.

Let $s > 0$ be arbitrary and $\iota > 0$ such that (7.109) and (7.110) hold. In the following we show that $\nabla_\theta \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M, M, (\rho^j)_j, \bar{C})$ with some suitable chosen $\bar{C} > 0$. It holds that

$$\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y, x, \theta) = \frac{\nabla_\theta(\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2}\Big(1 - y\frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2}\Big).$$

We have for $|\theta - \tilde{\theta}|_1 < \iota$:

$$\begin{aligned}
&2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x',\theta)|\\
\leq\ & |y| \cdot \Big[\frac{|\sigma(x,\tilde{\theta})^2 - \sigma(x',\tilde{\theta})^2|}{\sigma_{min}^2} + \frac{\sigma(x',\tilde{\theta})^2}{\sigma(x',\theta)^2\sigma_{min}^2}|\sigma(x,\theta)^2 - \sigma(x',\theta)^2|\Big] \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2}\\
&+\Big(1 + |y| \cdot \frac{\sigma(x',\tilde{\theta})^2}{\sigma(x',\theta)^2}\Big) \cdot \Big(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x',\theta)^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x',\theta)^2)|}{\sigma(x',\theta)^2\sigma_{min}^2}|\sigma(x,\theta)^2 - \sigma(x,\theta')^2|\Big).
\end{aligned}$$

Using (7.106) and (7.110), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x',\theta)| \leq \bar{C}(1 + |y|) \cdot |x - x'|_{(\rho^j)_j,1} \cdot |x|_{(\rho^{js})_j,s}^s. \tag{7.113}$$

We have for $|\theta - \tilde{\theta}|_1, |\theta' - \tilde{\theta}|_1 < \iota$:

$$\begin{aligned}
&2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta')|\\
\leq\ & |y| \cdot \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2\sigma_{min}^2}|\sigma(x,\theta)^2 - \sigma(x,\theta')^2| \cdot \frac{|\nabla_\theta(\sigma(x,\theta)^2)|}{\sigma(x,\theta)^2}\\
&+\Big(1 + |y| \cdot \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta')^2}\Big) \cdot \Big(\frac{|\nabla_\theta(\sigma(x,\theta)^2) - \nabla_\theta(\sigma(x,\theta')^2)|}{\sigma_{min}^2} + \frac{|\nabla_\theta(\sigma(x,\theta')^2)|}{\sigma(x,\theta')^2\sigma_{min}^2}|\sigma(x,\theta)^2 - \sigma(x,\theta')^2|\Big).
\end{aligned}$$

Using (7.107) and (7.110), we obtain (component-wise) with some suitably chosen $\bar{C} > 0$:

$$2|\nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta) - \nabla_\theta \tilde{\ell}_{\tilde{\theta}}(y,x,\theta')| \leq \bar{C}(1 + |y|) \cdot |\theta - \theta'|_1 \cdot R_{M,M}(1,x) \cdot |x|_{(\rho^{js})_j,s}^s. \tag{7.114}$$

We conclude from (7.113) and (7.114) that $\nabla_\theta \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M, M, (\rho^j)_j, \bar{C})$. The proof for $\nabla_\theta^2 \tilde{\ell}$ is similar in view of (7.106), (7.107) and (7.110) and therefore omitted.

Regarding Assumption 2.2, notice that from the explicit representation (7.98) and the geometric decay (7.97) uniformly in t we have that

$$\begin{aligned}
\partial_t \tilde{P}_i(t) &= \sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\Big(\prod_{0\leq j<l} M_{i-j}(t)\Big)\partial_t M_{i-l}(t)\Big(\prod_{l<j\leq k-1} M_{i-j}(t)\Big)a_{i-k}(t)\\
&\quad + \sum_{k=0}^{\infty}\Big(\prod_{j=0}^{k-1} M_{i-j}(t)\Big)\partial_t a_{i-k}(t)
\end{aligned}$$

exists a.s. and has q-th moments, so does its first component $\partial_t \tilde{Y}_t(t)$. Similar arguments that were used to prove (7.100) can be applied here and yield for $t, t' \in [0,1]$:

$$\|\partial_t \tilde{Y}_i(t) - \partial_t \tilde{Y}_i(t')\|_q \le C' \cdot |t - t'|,$$

with some constant $C' > 0$, i.e. Assumption 7.17(B3') is shown.

From (7.103) and $\sup_{\theta \in \Theta} \lambda_{max}(B(\theta)) < 1$, it follows that $x_i \mapsto \sigma(x,\theta)^2$ is differentiable for all $i \in \mathbb{N}$ and

$$\partial_{x_i} \nabla_\theta^k (\sigma(x,\theta)^2) = \nabla_\theta^k c_i(\theta), \quad k = 0,1,2.$$

Let $M' = 1$. Similar as above it can be seen that $\nabla_\theta^3 \tilde{\ell} \in \mathcal{H}_{s,\iota}^{mult}(M', M', (\rho^j)_j, \bar{C})$. Note that

$$\begin{aligned}
\partial_{x_i} \nabla_\theta^2 \tilde{\ell}_{\tilde{\theta}}(y,x,\theta) \quad = \quad & \Big[- \frac{\nabla_\theta c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta (\sigma(x,\theta)^2)^{\mathsf{T}}}{2\sigma(x,\theta)^2} - \frac{\nabla_\theta (2\sigma(x,\theta)^2)}{\sigma(x,\theta)^2} \cdot \frac{\nabla_\theta c_i(\theta)^{\mathsf{T}}}{\sigma(x,\theta)^2} \\
& + \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta (\sigma(x,\theta)^2) \nabla_\theta (\sigma(x,\theta)^2)^{\mathsf{T}}}{\sigma(x,\theta)^4} \\
& + \frac{\nabla_\theta^2 c_i(\theta)}{2\sigma(x,\theta)^2} - \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\nabla_\theta^2 (\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2} \Big] \cdot \Big(1 - y \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2}\Big) \\
& + \Big[- \frac{\nabla_\theta (\sigma(x,\theta)^2) \nabla_\theta (\sigma(x,\theta)^2)^{\mathsf{T}}}{2\sigma(x,\theta)^4} + \frac{\nabla_\theta^2 (\sigma(x,\theta)^2)}{2\sigma(x,\theta)^2} \Big] \cdot \frac{c_i(\theta)}{\sigma(x,\theta)^2} \frac{\sigma(x,\tilde{\theta})^2}{\sigma(x,\theta)^2} \cdot y \\
& + \text{similar terms (derivative of second summand)}
\end{aligned}$$

Each summand contains a factor $\nabla_\theta^k c_i(\theta)$ which is geometrically decaying by (7.110). Similar as above one can therefore see that $\nabla_\theta^2 \tilde{\ell}_{\tilde{\theta}} \in \mathcal{H}_{s,\iota}^{mult}(M', M', (\rho^j)_j, \bar{C}\rho^i)$. This shows Assumption 7.17(B2'). □

7.4.3. *Logistic Regression.*

Proof of Example 5.3. Define $\tilde{Y}_i(t) := \sum_{j=1}^m \mathbb{1}_{\{\zeta_{i,j} \le \pi(\tilde{X}_i(t)^{\mathsf{T}}\theta(t))\}}$. Put $M_y = 1$. The model follows (2.1) with $F_i(x,\theta) = \sum_{j=1}^m \mathbb{1}_{\{\xi_{i,j} \le \pi(x^{\mathsf{T}}\theta(i/n))\}}$. We have

$$\begin{aligned}
& \|F_i(x,\theta) - F_i(x',\theta)\|_1 \\
\le \quad & \sum_{j=1}^m \|\mathbb{1}_{\{\zeta_{i,j} \le \pi(x^{\mathsf{T}}\theta(t))\}} - \mathbb{1}_{\{\zeta_{i,j} \le \pi(x^{\mathsf{T}}\theta(t'))\}}\|_1 \\
\le \quad & m\{\mathbb{P}(\pi(x^{\mathsf{T}}\theta) \le \xi_{i,1} \le \pi((x')^{\mathsf{T}}\theta)) + \mathbb{P}(\pi((x')^{\mathsf{T}}\theta) \le \xi_{i,1} \le \pi(x^{\mathsf{T}}\theta))\} \\
\le \quad & 2m|\pi(x^{\mathsf{T}}\theta) - \pi((x')^{\mathsf{T}}\theta)|.
\end{aligned}$$

Since $|\partial_w \pi(w)| \le \frac{1}{4}$, we conclude that

$$\|F_i(x,\theta) - F_i(x',\theta)\|_1 \le \frac{m}{2} \sup_j \sup_{\theta\in\Theta} |\theta_j| \cdot |x - x'|_1,$$

i.e. (2.13) and thus Assumption 2.1(A5), (A7) is fulfilled.

Note that for fixed $c \in (0,1)$, $f(w) = \log(1+e^w) - c\cdot w$ is strongly convex with minimum at w_0 defined by $c = \frac{e^{w_0}}{1+e^{w_0}}$. It holds that

$$L(t,\theta) := \mathbb{E}\ell(\tilde{Y}_0(t), \tilde{X}_0(t), \theta) = m \cdot \mathbb{E}\Big[\log\Big(1 + \exp\big(\tilde{X}_0(t)^{\mathsf{T}}\theta\big)\Big) - \pi(\tilde{X}_0(t)^{\mathsf{T}}\theta(t)) \cdot \tilde{X}_0(t)^{\mathsf{T}}\theta\Big].$$

By a Taylor expansion of f around w_0, we obtain $f(w) = f(w_0) + \frac{1}{2}(w - w_0)^2 \partial_w f(\tilde{w})$. Since $f''(w) = \frac{e^w}{(1+e^w)^2}$ is increasing for $w < 0$ and decreasing for $w > 0$, $f''(\tilde{w}) \ge \min\{\pi(w), \pi(w_0)\}$. In the following we use the notation $|x|_A^2 := x^{\mathsf{T}}Ax$ for a weighted vector norm. We obtain that

$$L(t,\theta) - L(t,\theta(t)) \ge |\theta - \theta(t)|^2_{\tilde{V}(t,\theta)},$$

with $\tilde{V}(t,\theta) = \mathbb{E}\big[\min\{\pi(\tilde{X}_0(t)^{\mathsf{T}}\theta), \pi(\tilde{X}_0(t)^{\mathsf{T}}\theta(t))\}\tilde{X}_0\tilde{X}_0(t)^{\mathsf{T}}\big]$. If $\tilde{V}(t,\theta)$ was not positive definite for one θ, there would exist $v \in \mathbb{R}^p$ such that $v'\tilde{V}(t,\theta)v = 0$ which would imply that either $v'\tilde{X}_0(t) = 0$ a.s. or $\min\{\pi(\tilde{X}_0(t), \theta(t)), \pi(\tilde{X}_0(t), \theta)\} = 0$ a.s.. But it holds $\pi(\tilde{X}_0(t), \theta) \in (0,1)$ a.s. since $\sup_{j=1,\dots,p} |\tilde{X}_{0j}(t)| < \infty$ a.s. and Θ is compact. Furthermore, $v'\tilde{X}_0(t) = 0$ a.s. is a contradication to the positive definiteness of $\mathbb{E}[\tilde{X}_0(t)\tilde{X}_0(t)^{\mathsf{T}}]$. Thus $\tilde{V}(t,\theta)$ is positive definite for each θ and we conclude that $L(t,\theta)$ is uniquely minimized by $\theta = \theta(t)$. This shows Assumption 2.1(A3). We furthermore have

$$\begin{aligned}
\nabla_\theta \ell(z,\theta) &= m\pi(x^{\mathsf{T}}\theta)x - yx,\\
\nabla_\theta^2 \ell(z,\theta) &= m\frac{\exp(x^{\mathsf{T}}\theta)}{(1+\exp(x^{\mathsf{T}}\theta))^2} \cdot xx^{\mathsf{T}}.
\end{aligned}$$

It is easy to see that $\ell \in \mathcal{H}(1,1,\chi,\tilde{C})$, $\nabla_\theta \ell \in \mathcal{H}(1,2,\chi,\tilde{C})$ and $\nabla_\theta \ell \in \mathcal{H}(1,2,\chi,\tilde{C})$ with some $\tilde{C} > 0$ and $\chi = (1,\dots,1,0,0,\dots)$, a vector with p ones followed from zeros, i.e. Assumption 2.1(A1).

Since $\tilde{Y}_i(t)$ given $\tilde{X}_i(t)$ is binomial distributed with parameters $(m, \pi(\tilde{X}_0(t)^{\mathsf{T}}\theta(t)))$, we have

$$\mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t), \theta(t)) | \tilde{X}_0(t)] = m\pi(\tilde{X}_0(t)^{\mathsf{T}}\theta)\tilde{X}_0(t) - m\pi(\tilde{X}_0(t)^{\mathsf{T}}\theta)\tilde{X}_0(t) = 0.$$

Furthermore, $(\nabla_\theta \ell(\tilde{Z}_i(t), \theta(t)))_i$ is an uncorrelated sequence, thus we have $\Lambda(t) = I(t)$. Here,

$$V(t) = \mathbb{E}\nabla_\theta^2 \ell(\tilde{Z}_0(t), \theta(t)) = m\mathbb{E}\Big[\frac{\pi(\tilde{X}_0(t)^{\mathsf{T}}\theta(t))}{1 + \exp(\tilde{X}_0(t)^{\mathsf{T}}\theta(t))} \cdot \tilde{X}_0(t)^{\mathsf{T}}\tilde{X}_0(t)\Big],$$

which is positive definite by a similar argumentation as it was done for $\tilde{V}(t,\theta)$. Finally, since $\tilde{Y}_0(t)$ is binomial distributed with parameters $(m, \pi(\tilde{X}_0(t)^\mathsf{T}\theta(t)))$ given $\tilde{X}_0(t)$, we obtain

$$\begin{aligned} I(t) &= \mathbb{E}[\nabla_\theta \ell(\tilde{Z}_0(t),\theta(t))\nabla_\theta \ell(\tilde{Z}_0(t),\theta(t))^\mathsf{T}] \\ &= \mathbb{E}\big[\big(m\pi(\tilde{X}_0(t)^\mathsf{T}\theta(t)) - \tilde{Y}_0(t)\big)^2 \tilde{X}_0(t)\tilde{X}_0(t)^\mathsf{T}\big] \\ &= m\mathbb{E}[\pi(\tilde{X}_0(t)^\mathsf{T}\theta(t))(1-\pi(\tilde{X}_0(t)^\mathsf{T}\theta(t)))\tilde{X}_0(t)\tilde{X}_0(t)^\mathsf{T}] = V(t), \end{aligned}$$

and thus its positive definiteness and Assumption 2.1(A4).

□