Boosting Model-free predictions for econometric datasets

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January 6, 2021

Abstract

In this paper we propose some novel non-parametric prediction methods to perform short- and long-term aggregated forecasting for the log-returns of econometric datasets. The previous works in the regime of NoVaS model-free prediction are restricted to short-term forecasting only. Often practitioners and traders want to understand the future trend for a longer time into the future instead of just a single or a few steps ahead forecasts. This article serves two purposes. First it explores robustness of existing model-free methods for long-term predictions and then it introduces, with systematic justification, some new methods that improve the existing ones for both short- and long-term predictions. We provide detailed discussions of the existing and new methods and challenge the new ones with extensive simulations and real-life data that use pseudo-out-of-sample criteria. Finally, we show our methods can outperform existing state-of-art methods for short- and long-term forecasting where generally speaking, the improvement is considerable for long-term ones. One interesting feature of our methods is that it shows significant improvement compared to existing methods under strong volatile movements and shorter sample size.

Keywords: ARCH-GARCH, Model free, Aggregated forecasting

1 Introduction

Accurate and robust forecasting remains the most important goal that is achieved by analyzing econometric datasets. Short-term forecasting refers to the prediction methods that estimate a single or a few steps ahead values after observing a considerably large dataset from the past. However, in the scheme of a bigger picture the accuracy for just a single step might be inconsequential unless the predictive method can sustain its accuracy for a longer horizon in the future. In the time-series and econometrics literature such a practice is called long-term forecasting. In this paper we explore both short- and long-term forecasting for the volatility of financial returns using some existing model-free methods and build on some robust new methods that show significant improvement in prediction accuracy for a diverse range of simulated and real-life datasets.

During the prediction process, an inescapable intermediate process is building a model to describe historical data. Consequently, prediction results are restricted to this specific model used. However, the used model may not be correct within data, and the wrong model can even give better predictions sometimes (Politis, 2015). Therefore, it is hard as a practitioner to determine which model should be used in this intermediate stage. With the Model-free Prediction Principle being first proposed by Politis (2003), people can do predictions using data at hand directly without performing a model-fitting process. Applying this prediction principle one can remove the restriction of building a specific model and put all emphasis on data itself. Moreover, this principle possesses the potential to be embedded into different statistical models to further improve their performance (Sperlich, 2013).

Based on the insight of the Model-free Prediction Principle, a straightforward application is in the prediction of squared financial log-returns (i.e., equivalent to volatility forecast to some extent)[1]. The forecasting of volatility has become an interesting and important topic in the financial econometrics, such volatility forecasts can be used to manage financial activities and volatility itself also affects the forecast accuracy. (Engle and Patton, 2001; Du and Budescu, 2007). GARCH-type models' abilities to forecast the absolute magnitude and the quantiles or entire density of financial returns were shown by Engle and Patton (2001) using Dow Jones Industrial Index. Later, many studies about comparing the performance of different GARCH-type models on predicting volatility of financial series were conducted (Chortareas et al., 2011; González-Rivera et al., 2004; Herrera et al., 2018; Lim and Sek, 2013; Peters, 2001; Wilhelmsson, 2006; Zheng, 2012). Additionally, some researchers also tried to develop the GARCH model further, such as adopting smoothing parameter and adding more related information to estimate model (Breitung and Hafner, 2016; Chen et al., 2012; Fiszeder and Perczak, 2016; Taylor, 2004). In summary, although there are many different types of GARCH models, it is a difficult problem to determine which single GARCH model outperforms others uniformly since the performance of these models heavily depends on the error distribution, the length of prediction horizon and the property of datasets.

With the Model-free Prediction Principle being proposed, we acquire another powerful method–NoVaS method–to do volatility forecasting. The NoVaS method is applying the normalizing and variance-stabilizing transformation–herein referred to as NoVaS transformation–to do predictions (Politis, 2003, 2007, 2015). Within this method, an important model is the ARCH model which was proposed by Engle (1982):

$$Y_t = W_t\sqrt{a + \sum_{i=1}^{p} a_i Y_{t-i}^2} \tag{1.1}$$

In Eq. (1.1), the $a \geq 0$, $a_i \geq 0$, for all $i = 1, \cdots, p$; $W_t \sim i.i.d.N(0,1)$. The ARCH model is created for modeling the variance of a time series and plays a crucial role in the context of econometrics and finance. Later, we provide a thorough discussion on how to derive the NoVaS method based on the ARCH model. Some previous studies have shown that the NoVaS method possesses better predicting performance than GARCH-type models on forecasting squared financial returns, such as Gulay and Emec (2018) showed the NoVaS method could beat GARCH-type models (GARCH, EGARCH and GJR-GARCH) with generalized error distribution by comparing the pseudo-out of sample[2] (POOS) forecasting performance. Chen and Politis (2020) showed the "Time-varying" NoVaS method was robust against possible nonstationarities in the data. Furthermore, Chen and Politis (2019) extended this NoVaS approach to satisfy the demand of multi-step ahead predictions of volatility. They also verified this approach could avoid error accumulation problems for 5-steps ahead predictions and outperformed the standard GARCH model for most of time.

However, only 5-steps ahead forecasting using this method was performed so far. Practitioners typically are still interested in forecasting longer horizons (e.g., 30-steps ahead predictions). This type of forecasting plays a very important role in strategic decisions of fiscal policies (Ashiya, 2003; Bansal et al., 2016; Kitsul and Wright, 2013; Morikawa, 2019). Therefore, for fulfilling the inadequacy of the empirical evidence of NoVaS methods' usefulness on long-term predictions, this article wants to compare the abilities of NoVaS methods and GARCH-type methods on long-term forecasting of simulated data and real-life data. Moreover, to the best of our knowledge, previous studies in this regime mainly focus on comparing the NoVaS method with different GARCH-type models, and omit the potential of developing the NoVaS method itself further. Therefore, this article also attempts to build a novel NoVaS transformation which is based on the GARCH(1,1) model and also try to refine the structure of NoVaS transformation to further empower this method.

In this article, the long-term forecasting is presented as the aggregated multi-step ahead predictions, i.e., using the sliding-window mean of h-steps ahead predictions to be the forecast of the aggregated

[1] Squared returns are an unbiased but very noisy measure of volatility pointed out by Andersen and Bollerslev (1998). Additionally, Awartani and Corradi (2005) showed that using squared returns as a proxy for volatility can render a correct ranking of different GARCH models in terms of a quadratic loss function.

[2] The pseudo-out-of-sample forecasting analysis means using data up to and including current time to predict future values.

h-steps ahead value. This aggregated value can provide some inferences about future situations at the overall level and can also be seen as a measure to compare different methods' performance. Some aggregated prediction works have been done, such as time-aggregated predictions and Prediction Intervals (PIs) were applied to depict the future situation of electricity price or financial data (Chudỳ et al., 2020; Karmakar et al., 2020). Moreover, time-aggregated predictions were used to measure the performance of different types of GARCH methods (Fryzlewicz et al., 2008).

We now summarize our main findings and contributions in this paper. We explore the existing NoVaS method for how they compare to the standard GARCH based forecasting for long-term predictions and show that these methods are generally much better and robust to diverse range of datasets. Next, we propose two different tweaks to the existing NoVaS method resulting in three new methods. Finally we compare these NoVaS-type methods with the standard GARCH based forecasting and GARCH-bootstrap based methods. We use POOS criteria for measuring effectiveness of these different competing methods. Our simulations show that our methods are robust to different range of data generating processes. Even though our methods are based on the standard GARCH structure we show that they are also robust to model misspecification i.e. the performance is satisfactory even if the actual data comes from a different GARCH-type model. Moving on to real-datasets, we look different stock and index data for a old and new 2-year period. We also wanted to challenge our methods by seeing how our methods perform when the past data is not very long. Interestingly, we find out that improvements from our methods are amplified for shorter 1-year period predictions and this is also tested for multiple stock and index data. Finally, we are all aware how the econometric datasets went through a lot of structural changes due to the Covid-19 pandemic. We show our methods, generally speaking, bring significant improvement when used on the different classes of last years data from Nov 2019 to Oct 2020. This lets us conjecture that our methods can provide significant improvement for smaller samples and strong volatile movements. Additionally, the long-term forecasting methods of Chudỳ et al. (2020) and Karmakar et al. (2020) showed inadequate performance for short-term forecasting. In contrast, new methods in this article exhibit satisfied abilities for both short- and long-term forecasting, which implies these methods may be more adaptable.

The rest of the article is organized as follows. Details about the existing NoVaS transformation and motivations to propose a new transformation are explained in Section 2. In Section 3, we start by proposing our new methods and then introduce a novel a_0-removed technique. These methods are presented in algorithmic forms at the end of this section. For substantiating the usefulness of two proposals and the advantage of NoVaS-type methods on long-term forecasting, the POOS predictions of different methods on sixteen different simulated datasets are compared in Section 4. Additionally, we present a detailed and comprehensive analysis of real-world datasets to further substantiate our proposals in Section 5.

2 NoVaS method

In this section, details about the NoVaS method are given. We first introduce the Model-free Prediction Principle which is the core idea behind the NoVaS method. Then, we present how the NoVaS transformation can be built from an ARCH model. Finally, we propose two potential scopes of improvement within the existing NoVaS method, and then go on to build new NoVaS transformations.

2.1 Model-free prediction principle

Before presenting the NoVaS method in detail, we throw some light on the insight of Model-free Prediction Principle. The main idea behind this principle is applying an invertible transformation function H_n which can map the non-*i.i.d.* vector $\{Y_i\,; i = 1, \cdots, n\}$ to a vector $\{\epsilon_i;\ i = 1, \cdots, n\}$ that has *i.i.d.* components. Since the prediction of *i.i.d.* data is somewhat standard, the prediction of Y_{n+1} can be easily obtained by transforming the prediction of $\hat{\epsilon}_{n+1}$ back using H_n^{-1}. For example, we can express the prediction $\hat{Y}_{n+1}$ as a function of $\boldsymbol{Y}_n$, $\boldsymbol{X}_{n+1}$ and $\hat{\epsilon}_{n+1}$:

$$\hat{Y}_{n+1} = f_{n+1}(\boldsymbol{Y}_n, \boldsymbol{X}_{n+1}, \hat{\epsilon}_{n+1}) \tag{2.1}$$

Where, $\boldsymbol{Y}_n$ denotes all historical data $\{Y_t;\ i = 1, \cdots, n\}$; $\boldsymbol{X}_{n+1}$ is the collection of all predictors and it also contains the value of future predictor X_{n+1}. Although a qualified transformation H_n is hard to find for general case, we have naturally existing forms of H_n in some situations, such as in linear time series environment. In this article, we will show how to utilize existing forms–ARCH and GARCH models–to build NoVaS transformations to predict volatility of financial returns. One thing should be noted is that if we do not have these existing forms for some situations, more complicated procedure to obtain transformation H_n is needed (see (Politis, 2015) for more details).

After we acquire Eq. (2.1), we can even predict the general form of Y_{n+1} such as $g(Y_{n+1})$. The data-based optimal predictor of $g(Y_{n+1})$ under L_1 or L_2 criterion can be written as below:

$$\begin{aligned} g(Y_{n+1})_{L_1} &= \frac{1}{M}\sum_{m=1}^{M} g(f_{n+1}(\boldsymbol{Y}_n, \boldsymbol{X}_{n+1}, \hat{\epsilon}_{n+1}^{(m)})) \\ g(Y_{n+1})_{L_2} &= \text{Median of } \{g(f_{n+1}(\boldsymbol{Y}_n, \boldsymbol{X}_{n+1}, \hat{\epsilon}_{n+1}^{(m)}))\}_{m=1}^{M} \end{aligned} \tag{2.2}$$

In Eq. (2.2), $\hat{\epsilon}_{n+1}^{(m)}$ is generated from its own distribution by bootstrapping or Monte Carlo method (recall the $\{\epsilon_i\}$ are $i.i.d.$); M takes a large number (5000 in this article).

2.2 NoVaS transformation

The NoVaS transformation is a straightforward application of the Model-free Prediction Principle. More specifically, the NoVaS transformation is a qualified function H_n which is mentioned in Section 2.1. The starting point of building this transformation is the ARCH model described in Eq. (1.1) of Section 1. In other words, the structure of the ARCH model gives us a ready-made H_n which can potentially satisfy the requirement of Model-free Prediction Principle. For deriving the desired H_n, we can express the $\{W_t\}$ in Eq. (1.1) using other terms:

$$W_t = Y_t / \sqrt{a + \sum_{i=1}^{p} a_i Y_{t-i}^2} \quad \text{for } t = p+1, \cdots, n \tag{2.3}$$

Subsequently, the Eq. (2.3) can be seen as a potential form of H_n. However, some additional adjustments were made by Politis (2003). Firstly, the Y_t was added into the denominator in the right hand side of Eq. (2.3) for obeying the rule of causal estimate which is only using present and past data. Secondly, the constant a was replaced by αs_{t-1}^2 to create a scale invariant parameter α. Consequently, after observing the $\{Y_1, \cdots, Y_n\}$, the adjusted Eq. (2.3) can be written as Eq. (2.4):

$$W_t = \frac{Y_t}{\sqrt{\alpha s_{t-1}^2 + a_0 Y_t^2 + \sum_{i=1}^{p} a_i Y_{t-i}^2}} \quad \text{for } t = p+1, \cdots, n \tag{2.4}$$

In Eq. (2.4), $\{Y_t;\ t = 1, \cdots, n\}$ is the targeted data, such as financial returns in this paper; the $\{W_t;\ t = p+1, \cdots, n\}$ can be seen as the transformed $\boldsymbol{\epsilon}_n$ from the $\{Y_t\}$ based on the Model-free Prediction Principle's notations; the α is a fixed scale invariant constant; The s_{t-1}^2 is an estimator of the variance of $\{Y_i;\ i = 1, \cdots, t-1\}$ and can be calculated by $(t-1)^{-1}\sum_{i=1}^{t-1}(Y_i - \mu)^2$, with μ being the mean of $\{Y_i;\ i = 1, \cdots, t-1\}$.

The observed residuals $\{W_t;\ t = p+1, \cdots, n\}$ expressed in Eq. (2.4) are assumed to be $i.i.d.N(0,1)$, but it is not true so far. For making Eq. (2.4) be a qualified function H_n (i.e., making $\{W_t\}$ really obey standard normal distribution), we still need to add some restrictions on α and $a_0, \cdots, a_p$. Recalling the NoVaS transformation means normalizing and variance-stabilizing transformation, we first stabilize the variance. We do this by requiring:

$$\alpha \geq 0, a_i \geq 0 \text{ for all } i \geq 0, \alpha + \sum_{i}^{p} a_i = 1 \tag{2.5}$$

By requiring unknown parameters in Eq. (2.4) to satisfy the requirement Eq. (2.5), we can make the $\{W_t\}$ series possess approximate unit variance. Additionally, α and $a_0, \cdots, a_p$ need to be selected carefully to achieve same variance of $\{W_t\}$, i.e., we still need to take p small enough, as well as α small enough or even equal to zero (Chen and Politis, 2019). In the work of Politis (2015), two different structures of $a_0, \cdots, a_p$ were provided:

$$\begin{aligned}
&\text{Simple NoVaS: } \alpha = 0, a_i = \frac{1}{p+1} \text{ for all } 0 \leq i \leq p \\
&\text{Exponential NoVaS: } \alpha = 0, a_i = c'e^{-ci} \text{ for all } 0 \leq i \leq p,\ c' = \frac{1}{\sum_{i=0}^{p} e^{-ci}}
\end{aligned} \tag{2.6}$$

We use S-NoVaS and E-NoVaS to denote these two NoVaS methods in Eq. (2.6). For the S-NoVaS, all a_i taking same value means we assign same weights on past data. Similarly, for the E-NoVaS, the $a_0, \cdots, a_p$ are exponentially positive decayed coefficients, which means we assign decayed weights on past data. Note that α is equal to 0 in both methods above. If we allow α takes positive small value, we can get two different methods:

$$\begin{aligned}
&\text{Generalized Simple NoVaS: } \alpha \neq 0, a_i = \frac{1-\alpha}{p+1} \text{ for all } 0 \leq i \leq p \\
&\text{Generalized Exponential NoVaS: } \alpha \neq 0, a_i = c'e^{-ci} \text{ for all } 0 \leq i \leq p,\ c' = \frac{1-\alpha}{\sum_{i=0}^{p} e^{-ci}}
\end{aligned} \tag{2.7}$$

We use GS-NoVaS and GE-NoVaS to denote these two generalized NoVaS methods in Eq. (2.7). The α in both generalized methods takes value from $(0, 1)$[3]. Obviously, NoVaS and generalized NoVaS methods all meet the requirement of Eq. (2.5).

Additionally, we still need to make the $\{W_t\}$ independent. In practice, the $\{W_t\}$ transformed from financial returns is usually uncorrelated[4]. Therefore, we just need to make the empirical distribution of $\{W_t\}$ close to the standard normal distribution (i.e., normalizing $\{W_t\}$). Note that the distribution of financial returns is usually symmetric, thus, the kurtosis can serve as a simple distance measuring the departure of a non-skewed dataset from the standard normal distribution whose kurtosis is 3 (Politis, 2015). We use $\hat{F}_w$ to denote the empirical distribution of $\{W_t\}$ and use $KURT(W_t)$ to denote the kurtosis of $\hat{F}_w$. Then, for making $\hat{F}_w$ close to the standard normal distribution, we can minimize $|KURT(W_t) - 3|$[5] to obtain the optimal combination of $\alpha, a_1, \cdots, a_p$. Consequently, the NoVaS transformation is determined.

From the thesis of Chen (2018), the Generalized NoVaS methods are better for interval prediction of squared returns and estimation of volatility than other NoVaS methods. Additionally, the GE-NoVaS method which assigns exponentially decayed weight to the past data is more reasonable than the GS-NoVaS method which handles past data equally. Therefore, in this article, we verify the advantage of our new methods by comparing them with the GE-NoVaS method. Before going further to propose the new NoVaS transformation, we talk more details about the GE-NoVaS method and our motivations to create new methods.

2.3 GE-NoVaS method

For the GE-NoVaS method, the fixed α is larger than 0 and selected from a grid of possible values based on predictions performance. In this article, we define this grid as $(0.1, 0.2, \cdots, 0.8)$ which contains 8 discrete values[6]. From Section 2.2, based on the guide of Model-free Prediction Principle, we already get the function H_n of GE-NoVaS method by requiring parameters to satisfy the requirement of Eq. (2.5) and minimizing $|KURT(W_t) - 3|$. For completing the model-free prediction process, we still need to figure out the form of H_n^{-1}. Through Eq. (2.4), the H_n^{-1} can be written as follows:

$$Y_t = \sqrt{\frac{W_t^2}{1 - a_0 W_t^2}(\alpha s_{t-1}^2 + \sum_{i=1}^{p} a_i Y_{t-i}^2)} \text{ for } t = p+1, \cdots, n \tag{2.8}$$

[3] If $\alpha = 1$, all a_i will equal to 0. It means we just standardize $\{Y_t\}$ and do not utilize the structure of ARCH model.

[4] If the $\{W_t\}$ is correlated, some additional adjustments need to be done, more details can be found in (Politis, 2015).

[5] More details about this minimizing process can be found in Politis (2015).

[6] People can refine this grid to get a better transformation. However, computation complexity will also improve.

Based on Eq. (2.8), it is easy to get the analytical form of Y_{n+1}, which can be expressed as below:

$$Y_{n+1} = \sqrt{\frac{W_{n+1}^2}{1 - a_0 W_{n+1}^2}(\alpha s_n^2 + \sum_{i=1}^{p} a_i Y_{n+1-i}^2)} \tag{2.9}$$

In Eq. (2.9), s_n^2 is an estimator of the variance of $\{Y_i;\ i = 1, \cdots, n\}$ and can be calculated by $n^{-1}\sum_{i=1}^{n}(Y_i - \mu)^2$, μ is the mean within data; the W_{n+1} comes from the empirical distribution $\hat{F}_w$ or the trimmed standard normal distribution[7]. Obviously, Y_{n+1} can be presented as a function of W_{n+1} and $Y_1, \cdots, Y_n$ as below:

$$Y_{n+1} = f_{GE}(W_{n+1}; Y_1, \cdots, Y_n) \tag{2.10}$$

For reminding us this relationship between Y_{n+1} and $W_{n+1}, Y_1, \cdots, Y_n$ is derived from the GE-NoVaS method, we use $f_{GE}(\cdot)$ to denote this function. It is not hard to find the Y_{n+h} can be expressed as:

$$Y_{n+h} = f_{GE}(W_{n+1}, \cdots, W_{n+h}, Y_1, \cdots, Y_n) \tag{2.11}$$

Similarly, $\{W_{n+1}, \cdots, W_{n+h}\}$ comes from the $\hat{F}_w$ or the trimmed standard normal distribution. Additionally, the historical data $Y_1, \cdots, Y_n$ are known, then Eq. (2.11) can be simplified to:

$$Y_{n+h} = f_{GE}(W_{n+1}, \cdots, W_{n+h}) \text{ for any } h \geq 1 \tag{2.12}$$

So far, we obtain the analytical form of Y_{n+h} from the GE-NoVaS transformation. The h-step ahead value only depends on *i.i.d.* $\{W_{n+1}, \cdots, W_{n+h}\}$.

In the algorithm of performing the GE-NoVaS method, $\{W_{n+1}^*, \cdots, W_{n+h}^*\}$ are generated M times from a trimmed standard normal distribution by Monte Carlo method or bootstrapped from its empirical distribution $\hat{F}_w$. Then, plug these $\{W_{n+1}^{*,(m)}, \cdots, W_{n+h}^{*,(m)}\}_{m=1}^{M}$ into the Eq. (2.12), M pseudo predictions $\hat{Y}_{n+h}$ are obtained. Recording all $\{\hat{Y}_{n+h}^{(1)}, \cdots, \hat{Y}_{n+h}^{(M)}\}$, the L_1 and L_2 risk optimal predictors of Y_{n+h} are the sample median and mean of $\{\hat{Y}_{n+h}^{(1)}, \cdots, \hat{Y}_{n+h}^{(M)}\}$, respectively. Similar with Eq. (2.2), we can even predict the general form of Y_{n+h}, such as $g(Y_{n+h})$.

2.4 Motivations of building new NoVaS transformation

In last few sections, we illustrated the procedure of using the GE-NoVaS method to calculate L_1 and L_2 predictor of Y_{n+h}. However, the form of coefficients $a_1, \cdots, a_p$ in the Eq. (2.4) is somewhat arbitrary. Note that the GE-NoVaS method simply sets $a_1, \cdots, a_p$ to be exponentially decayed. However, a more regular form of $a_1, \cdots, a_p$ should be built based on the ARCH model itself without assigning any form on coefficients. Therefore, a new approach to determine the form of $a_1, \cdots, a_p$ based on the GARCH(1,1) model is proposed. Subsequently, the GARCH-NoVaS (GA-NoVaS) transformation is built.

Additionally, Chen and Politis (2019) showed this approach can avoid the error accumulation problem which is derived from the traditional multi-stage prediction, i.e., using predicted values to predict further data iteratively. However, this article will show it still renders extremely large aggregated multi-step ahead prediction under L_2 risk measure sometimes . The reason for this phenomenon is the denominator of Eq. (2.9) will be quite small when the generated W^* is very close to the $\sqrt{1/a_0}$. In this situation, the prediction error will be amplified. Moreover, when the long-step ahead prediction is desired, this amplification will be accumulated and final prediction will be dampened. Therefore, this article makes another proposal which is refining the NoVaS transformation structure by removing the a_0 term. Moreover, this article will show this adjustment is not only empirically and theoretically reasonable, but also beneficial to the accuracy of predictions and even slightly useful to reduce computational complexity.

[7] More details about the meaning of trimmed standard normal distribution are presented in Section 3.2.

3 Our new methods

In this section, we first propose the GA-NoVaS method which is developed from the GARCH(1,1) model directly without assigning any specific form of $a_1, \cdots, a_p$. Then, the reason of the a_0 term being potentially problematic is explained in detail. The evidence of the problematic issue of a_0 is also provided. Finally, two Algorithms of new methods are given.

3.1 The GA-NoVaS transformation

Recall the GE-NoVaS method mentioned in Section 2.3, it was built by taking advantage of the ARCH(p) model, the p takes initially large value in the algorithm of performing the GE-NoVaS. However, assigning one specific form to $a_1, \cdots, a_p$ is not so convincing. Thus, we try to use another approach to build $a_1, \cdots, a_p$ directly without assigning any prior form to these parameters. We call this NoVaS transformation method by GA-NoVaS. The starting point of building this transformation is the ARCH(∞) model. Note that the equivalent representation of the ARCH(∞) model is the GARCH(1,1) model, the proof of this is trivial, see (Politis and McElroy, 2019) for some references. Thus, based on the GARCH(1,1) model, we attempt to build a more reasonable approach to obtain the form of $a_1, \cdots, a_p$.

We present the GARCH(1,1) model as Eq. (3.1):

$$\begin{aligned} Y_t &= \sigma_t W_t \\ \sigma_t^2 &= a + a_1 Y_{t-1}^2 + b_1 \sigma_{t-1}^2 \end{aligned} \tag{3.1}$$

In Eq. (3.1), $a \geq 0$, $a_1 > 0$, $b_1 > 0$, and $W_t \sim i.i.d.N(0,1)$. Take Eq. (3.1) as the starting point to build the GA-NoVaS transformation. We first calculate expressions of $\sigma_{t-1}^2, \sigma_{t-2}^2, \cdots$ as follow:

$$\begin{aligned} \sigma_{t-1}^2 &= a + a_1 Y_{t-2}^2 + b_1 \sigma_{t-2}^2 \\ \sigma_{t-2}^2 &= a + a_1 Y_{t-3}^2 + b_1 \sigma_{t-3}^2 \\ &\vdots \end{aligned} \tag{3.2}$$

Plug all components in Eq. (3.2) into Eq. (3.1), one equation sequence can be gotten:

$$\begin{aligned} Y_t &= W_t \sqrt{a + a_1 Y_{t-1}^2 + b_1 \sigma_{t-1}^2} \\ &= W_t \sqrt{a + a_1 Y_{t-1}^2 + b_1 (a + a_1 Y_{t-2}^2 + b_1 \sigma_{t-2}^2)} \\ &= W_t \sqrt{a + a_1 Y_{t-1}^2 + b_1 a + b_1 a_1 Y_{t-2}^2 + b_1^2 (a + a_1 Y_{t-3}^2 + b_1 \sigma_{t-3}^2)} \\ &\vdots \end{aligned} \tag{3.3}$$

Iterating the process in Eq. (3.3), the limiting form of Y_t can be written as Eq. (3.4):

$$Y_t = W_t \sqrt{\sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2 + \sum_{j=0}^{\infty} a b_1^j} = W_t \sqrt{\sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2 + \frac{a}{1-b_1}} \tag{3.4}$$

Recall the sum of a_1 and b_1 is required to less than 1 for stationarity of this model. Furthermore, we can rewrite Eq. (3.4) to get a potential function H_n which is corresponding to the GA-NoVaS method:

$$W_t = \frac{Y_t}{\sqrt{\sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2 + \frac{a}{1-b_1}}} \tag{3.5}$$

Recall the adjustment adopted in the existing GE-NoVaS method, the total difference between Eqs. (2.3) and (2.4) can be seen as the term a being replaced by $\alpha s_{t-1}^2 + a_0 Y_t^2$. Apply this same adjustment to

Eq. (3.5), then this equation will be changed to the form as follows:

$$W_t = \frac{Y_t}{\sqrt{\frac{a_0 Y_t^2 + \alpha s_{t-1}^2}{1-b_1} + \sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2}} = \frac{Y_t}{\sqrt{\frac{a_0 Y_t^2}{1-b_1} + \frac{\alpha s_{t-1}^2}{1-b_1} + \sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2}} \tag{3.6}$$

In Eq. (3.6), since $\alpha/(1-b_1)$ is also required to take small positive value, this term can be seen as a β ($\beta \geq 0$) which is equivalent with the α in the existing GE-NoVaS method. Thus, we can simplify $\alpha s_{t-1}^2/(1-b_1)$ to βs_{t-1}^2. For keeping the same notation with the GE-NoVaS method, we use αs_{t-1}^2 to represent $\alpha s_{t-1}^2/(1-b_1)$. Then Eq. (3.6) can be represented as:

$$W_t = \frac{Y_t}{\sqrt{\frac{a_0 Y_t^2}{1-b_1} + \alpha s_{t-1}^2 + \sum_{i=1}^{\infty} a_1 b_1^{i-1} Y_{t-i}^2}} \tag{3.7}$$

For getting a qualified GA-NoVaS transformation, we still need to make the function H_n (i.e., Eq. (3.7)) satisfy the requirement of the Model-free Prediction Principle. Recall that in the existing GE-NoVaS method, $\alpha + \sum_{i=0}^{p} a_i$ in Eq. (2.4) is restricted to be 1 for meeting the requirement of variance-stabilizing and the optimal combination of $\alpha, a_1, \cdots, a_p$ is selected to make the empirical distribution of $\{W_t\}$ as close to the standard normal distribution as possible (i.e., minimizing $|KURT(W_t) - 3|$). Similarly, for getting a qualified H_n from Eq. (3.7), we require:

$$\frac{a_0}{1-b_1} + \alpha + \sum_{i=1}^{\infty} a_1 b_1^{i-1} = 1 \tag{3.8}$$

Under this requirement, since a_1 and b_1 are both less than 1, the $a_1 b_1^{i-1}$ will converge to 0 as the i converges to ∞, i.e., the $a_1 b_1^{i-1}$ is neglectable when i takes large value. So it is reasonable to replace $\sum_{i=1}^{\infty} a_1 b_1^{i-1}$ in Eq. (3.8) by $\sum_{i=1}^{q} a_1 b_1^{i-1}$, where q takes a large value. Then a truncated form of Eq. (3.7) can be written as Eq. (3.9):

$$W_t = \frac{Y_t}{\sqrt{\frac{a_0 Y_t^2}{1-b_1} + \alpha s_{t-1}^2 + \sum_{i=1}^{q} a_1 b_1^{i-1} Y_{t-i}^2}} \quad \text{for } t = p+1, \cdots, n \tag{3.9}$$

Now, we take the Eq. (3.9) as a potential function H_n. Then, the requirement of variance-stabilizing is changed to:

$$\frac{a_0}{1-b_1} + \alpha + \sum_{i=1}^{q} a_1 b_1^{i-1} = 1 \tag{3.10}$$

Additionally, for achieving the aim of normalizing, we fix the α to be one specific value from $\{0.1, 0.2, \cdots, 0.8\}$, and then search the optimal combination of α, a_0, a_1, b_1 from three grids of possible values of a_0, a_1, b_1[8] to minimize $|KURT(W_t) - 3|$. After getting a qualified H_n, the H_n^{-1} will be obtained immediately:

$$Y_t = \sqrt{\frac{W_t^2 (1-b_1)(\alpha s_{t-1}^2 + \sum_{i=1}^{q} a_1 b_1^{i-1} Y_{t-i}^2)}{1 - b_1 - W_t^2 a_0}} \quad \text{for } t = p+1, \cdots, n \tag{3.11}$$

Based on Eq. (3.11), the Y_{n+1} can be expressed as the equation follows:

$$Y_{n+1} = \sqrt{\frac{W_{n+1}^2 (1-b_1)(\alpha s_n^2 + \sum_{i=1}^{q} a_1 b_1^{i-1} Y_{n+1-i}^2)}{1 - b_1 - W_{n+1}^2 a_0}} \tag{3.12}$$

Also, it is not hard to express Y_{n+1} as a function of $W_{n+1}, \cdots, W_{n+h}$ and $Y_1, \cdots, Y_n$. Since $Y_1, \cdots, Y_n$ are known, we can write Y_{n+1} as a function of $W_{n+1}, \cdots, W_{n+h}$ like we did in Section 2.3:

$$Y_{n+h} = f_{GA}(W_{n+1}, \cdots, W_{n+h}) \quad \text{for any } h \geq 1 \tag{3.13}$$

[8] Each grid of possible values has range [0,1]. The sum of a_0, a_1, b_1 is required to less than 1. Additionally, we require the $a_0/(1-b_1)$ is the largest coefficient among all coefficients of Y^2 terms.

Once the expression of Y_{n+h} is figured out, we can apply the same procedure within the GE-NoVaS method to get optimal predictor of Y_{n+h} based on L_1 or L_2 risk criterion. To deal with α, we still adopt the same strategy used in the GE-NoVaS method, i.e., select optimal α from a grid of possible values based on prediction performance. One thing should be noticed is that the value of α is invariant during the process of predicting once we fix it as a specific value. More details about the algorithm of this new method can be found in Section 3.3.

3.2 The potentially problematic a_0

In this subsection, we give the explanation on why the a_0 term in Eq. (2.9) is problematic. The reason for this has been briefly discussed in Section 2.4. In this section, a more detailed explanation and empirical verification will be given.

For illustration, we focus on analyzing the GE-NoVaS method. Based on the model-free Prediction Principle. The related H_n and H_n^{-1} of the GE-NoVaS method are Eqs. (2.4) and (2.8), respectively. The potential problem of a_0 is derived from the H_n^{-1}, i.e., the process of transferring W_t back to the Y_t. See H_n and H_n^{-1} corresponding to the GE-NoVaS as follow:

$$W_t = \frac{Y_t}{\sqrt{\alpha s_{t-1}^2 + a_0 Y_t^2 + \sum_{i=1}^{p} a_i Y_{t-i}^2}} \;;\; Y_t = \sqrt{\frac{W_t^2}{1 - a_0 W_t^2}(\alpha s_{t-1}^2 + \sum_{i=1}^{p} a_i Y_{t-i}^2)} \;;\; \text{for } t = p+1, \cdots, n \tag{3.14}$$

From the first equation in Eq. (3.14), we can clearly find:

$$|W_t| \leq 1/\sqrt{a_0} \tag{3.15}$$

Recall our goal is obtaining *i.i.d.* $\{W_t\}$ which obeys standard normal distribution, so $\{W_t\}$ should have enough large range, otherwise the normality will be impaired. We can require $1/\sqrt{a_0} \geq 3$ because 99.7% of the mass of the standard normal distribution is found in the range ±3. Moreover, if we generate $\{W_{n+1}^*, \cdots, W_{n+h}^*\}$ from the normal distribution to do h-steps ahead predictions, it is necessary to use the standard normal distribution truncated to the range $\pm 1/\sqrt{a_0}$ for guaranteeing that Eq. (3.15) is satisfied (Chen and Politis, 2019). However, when the generated W^* is close to $1/\sqrt{a_0}$, this denominator can still be tiny even if with small probability. Then, the prediction results will be amplified and this amplification effect will be accumulated when we do long steps predictions. Consequently, the L_2 predictor of the future value will become unstable.

For illustrating this potential amplification effect which comes from the existence of a_0 term, we wanted to perform real-data analysis to show this phenomenon. We take 500 stock price data points of Microsoft Corporation from January 8, 1998 to December 31,1999 to do tests. First, we use these 500 price data to create 499 financial returns by the formula below:

$$Y_t = 100 \times log(X_{t+1}/X_t) \;\; \text{for } t = 1, \cdots, 499 \tag{3.16}$$

In Eq. (3.16), the X_t is the stock price and the Y_t is the financial return.

Then we use first 250 financial returns to do POOS 1-step, 5-steps and 30-steps ahead time aggregated predictions of squared financial returns under L_2 criterion. For exploring the prediction performance of the GE-NoVaS method on the whole dataset, we roll the 250 data points window through this dataset, i.e., use $\{Y_1, \cdots, Y_{250}\}$ to predict Y_{251}^2, $\{Y_{251}^2, \cdots, Y_{255}^2\}$ and $\{Y_{251}^2, \cdots, Y_{280}^2\}$, respectively; then use $\{Y_2, \cdots, Y_{251}\}$ to predict Y_{252}^2, $\{Y_{252}^2, \cdots, Y_{256}^2\}$ and $\{Y_{252}^2, \cdots, Y_{281}^2\}$, for 1-step, 5-steps and 30-steps respectively, and so on. Additionally, we define $\bar{Y}_{k,1}^2$, $\bar{Y}_{i,5}^2$ and $\bar{Y}_{j,30}^2$ to represent the time aggregated predictions of 1-step, 5-steps and 30-steps ahead financial squared returns, respectively:

$$\begin{aligned}
\bar{Y}_{k,1}^2 &= \hat{Y}_{k+1}^2, \; k = 250, \cdots, 498 \\
\bar{Y}_{i,5}^2 &= \frac{1}{5}\sum_{n=1}^{5} \hat{Y}_{i+n}^2, \; i = 250, \cdots, 494 \\
\bar{Y}_{j,30}^2 &= \frac{1}{30}\sum_{n=1}^{30} \hat{Y}_{j+n}^2, \; j = 250, \cdots, 469
\end{aligned} \tag{3.17}$$

In Eq. (3.17), the $\hat{Y}_t^2$ is the prediction of Y_t^2, t represents any future time point, such as $k+1, i+n$ or $j+n$. By rolling the prediction window through the whole dataset, we perform 249, 245 and 220 aggregated predictions for 1-step, 5-steps ahead and 30-steps ahead settings, respectively. For measuring the performance of the GE-NoVaS using these different prediction steps, we compare these predictions with true values which are calculated by realized financial squared returns, i.e., computing the performance value P by the following formula:

$$P_l = \sum_l (\bar{Y}^2_{l,h} - \sum_{n=1}^{h} (Y^2_{l+n}/h)) \tag{3.18}$$

In Eq. (3.18), the Y^2_{l+n} is the realized squared financial return; h represents three different prediction lengths and $l \in \{k, i, j\}$ In addition to compute numerical values, we can also plot time aggregated prediction results with true values to get a vivid presentation.

Fig. 1 depicts the performance of GE-NoVaS on predicting 1-step, 5-steps and 30-steps ahead time aggregated squared financial returns under L_2 risk criterion. Without loss of generality, we take α as 0.5. It is clearly found that there are a few extreme outliers in the rightmost subfigure which presents results of the GE-NoVaS on predicting 30-steps ahead time aggregated financial squared returns. This phenomenon is consistent with our supposition that the amplification effect created by the a_0 term may dampen the performance of GE-NoVaS when long term predictions are sought. On the other hand, this accumulated amplification effect is not severe in the first two subfigures of Fig. 1.

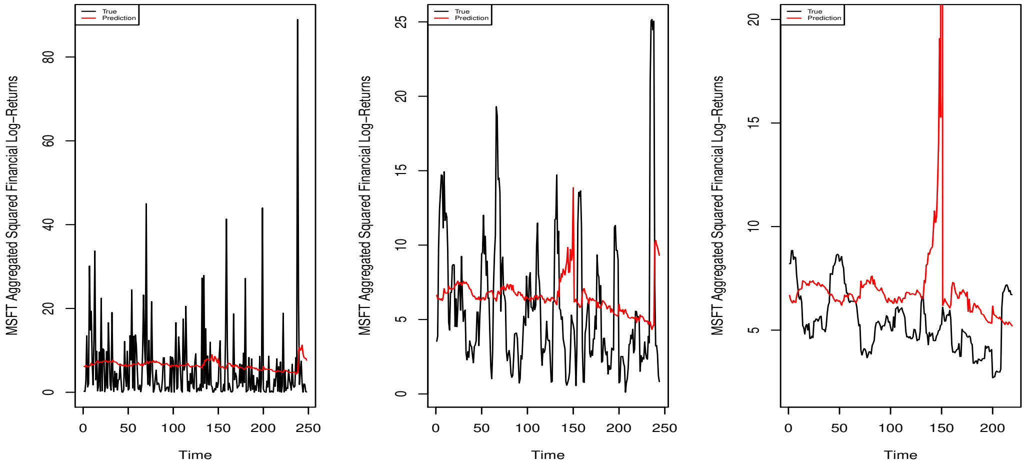

Figure 1: Curves of the true and predicted time aggregated squared financial returns. The predicted values are obtained from applying the GE-NoVaS method with α being 0.5 under three different lengths of prediction steps.

With this potential problem derived from the a_0 term, we put forth the idea of removing the a_0 term. Subsequently, with simple calculation, function H_n and H_n^{-1} corresponding to the GE-NoVaS-without-a_0 method can be presented as follow:

$$W_t = \frac{Y_t}{\sqrt{\alpha s^2_{t-1} + \sum_{i=1}^{p} a_i Y^2_{t-i}}} \; ; \; Y_t = \sqrt{W_t^2(\alpha s^2_{t-1} + \sum_{i=1}^{p} a_i Y^2_{t-i})} \; ; \text{ for } t = p+1, \cdots, n \tag{3.19}$$

Two equations in Eq. (3.19) still need to satisfy the requirement of normalizing and variance-stabilizing transformation. Therefore, we restrict $\alpha + \sum_{i=1}^{p} a_i = 1$ and still select the optimal combination of

$\alpha, a_1, \cdots, a_p$ by minimizing $|KURT(W_t) - 3|$. Then, Y_{n+1} can be expressed by Eq. (3.20):

$$Y_{n+1} = \sqrt{W_{n+1}^2(\alpha s_n^2 + \sum_{i=1}^{p} a_i Y_{n+1-i}^2)} \tag{3.20}$$

Remark 1. *Even though we do not include the effect of Y_t when we build the H_n, the expression of Y_{n+1} still contains the current value Y_n. It means the GE-NoVaS-without-a_0 method does not disobey the rule of causal prediction.*

Using the GE-NoVaS-without-a_0 method to do the same prediction process as before, we present Fig. 2 which depicts curves of prediction results and true values. We find there is no extreme outlier in the rightmost subfigure of Fig. 2. Moreover, from other two subfigures of Fig. 2, the predictions derived from the GE-NoVaS-without-a_0 method can better capture the features of true values for the 1-step and 5-steps ahead predictions.

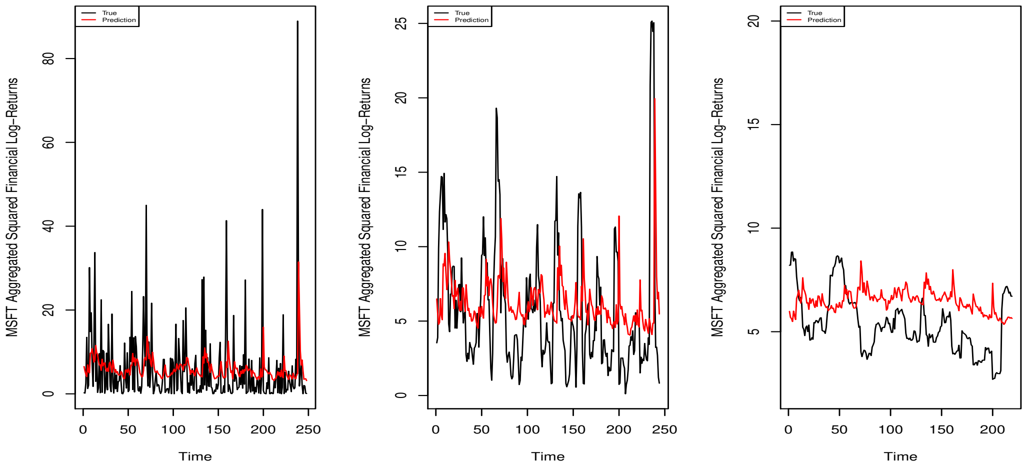

Figure 2: Curves of the true and predicted time aggregated squared financial returns. The predicted values are obtained from applying the GE-NoVaS-without-a_0 method with α being 0.5 under three different lengths of prediction steps.

Similarly, our proposed GA-NoVaS method can also be offered in a different variant without a_0 term. Eqs. (3.9) and (3.11) without a_0 term can be represented by following equations:

$$W_t = \frac{Y_t}{\sqrt{\alpha s_{t-1}^2 + \sum_{i=1}^{p} a_1 b_1^{i-1} Y_{t-i}^2}} \ ; \ Y_t = \sqrt{W_t^2(\alpha s_{t-1}^2 + \sum_{i=1}^{p} a_1 b_1^{i-1} Y_{t-i}^2)} \ ; \ \text{for } t = p+1, \cdots, n \tag{3.21}$$

$\alpha + \sum_{i=1}^{p} a_1 b_1^{i-1} = 1$ is required to satisfy the variance-stabilizing requirement and the optimal combination of α, a_1, b_1 is selected by minimizing $|KURT(W_t) - 3|$ to satisfy the normalizing requirement.

Similarly, for GE-NoVaS- and GA-NoVaS-without-a_0 methods, we can still express Y_{n+h} as a function of $\{W_{n+1}, \cdots, W_{n+h}\}$ and repeat the same procedure we did before to get L_1 and L_2 predictors. For example, we can derive the expression of Y_{n+h} using the GA-NoVaS-without-a_0 method:

$$Y_{n+h} = f_{\text{GA-without-}a_0}(W_{n+1}, \cdots, W_{n+h}) \ \text{ for any } h \geq 1 \tag{3.22}$$

Remark 2 (Slight computational efficiency of removing a_0)**.** *Note that the suggestion of removing a_0 can also lead a less time-complexity of the existing GE-NoVaS method and proposed forecasting algorithms.*

The reason for this phenomenon is simple. Recall the $1/\sqrt{a_0}$ is required to larger or equal to 3 for making $\{W_t\}$ have enough large range, i.e., a_0 is required to be less or equal to 0.111. However, the optimal combination of $\alpha, a_0, a_1, \cdots, a_p$ may not render a suitable a_0. For this situation, we need to increase the order p and repeat the normalizing and variance-stabilizing process till the a_0 in the optimal combination of $\alpha, a_0, a_1, \cdots, a_p$ is suitable. This repeating process definitely increases the computation workload.

3.3 Algorithms of new methods

In the two Sections 3.1 and 3.2, we presented the GA-NoVaS method and explained the reason of removing a_0 term. Subsequently, based on these two proposals, three new methods (GE-NoVaS-without-a_0, GA-NoVaS and GA-NoVaS-without-a_0) are created. In this section, we provide algorithms related to GA-NoVaS-type methods.

Let us consider the algorithm of the GA-NoVaS method. As we know, this method is one kind of NoVaS method and is developed from the GARCH(1,1) model. Unknown parameters a_0, a_1, b_1 are selected from three grids of possible values to normalize the $\{W_t;\ t = p+1, \cdots, n\}$ in Eq. (3.9). If our goal is the h-step ahead prediction of $g(Y_{n+h})$ using past $\{Y_t;\ t = 1, \cdots, n\}$, the algorithm of the GA-NoVaS method is described in Algorithm 1.

Algorithm 1: the h-step ahead prediction for the GA-NoVaS method

```
Step 1  Define a grid of possible $\alpha$ values, $\{\alpha_k;\ k = 1, \cdots, K\}$, three grids of possible $a_0$, $a_1$, $b_1$ values. Fix $\alpha = \alpha_k$, then calculate the optimal combination of $\alpha_k, a_0, a_1, b_1$ of the GA-NoVaS method.
Step 2  Derive the analytic form of Eq. (3.13) using the $\{a_0, a_1, b_1, \alpha_k\}$ from the first step.
Step 3  Generate $\{W^*_{n+1}, \cdots, W^*_{n+h}\}$ M times by the Monte Carlo method from a trimmed standard normal distribution or bootstrapping from $\hat{F}_w$. Plug these $\{W^*_{n+1}, \cdots, W^*_{n+h}\}$ into the analytic form of Eq. (3.13) to obtain M pseudo-values $\{\hat{Y}^{(1)}_{n+h}, \cdots, \hat{Y}^{(M)}_{n+h}\}$.
Step 4  Calculate the optimal predictor of $g(Y_{n+h})$ by taking the sample mean (under $L_2$ risk criterion) or sample median (under $L_1$ risk criterion) of the set $\{g(\hat{Y}^{(1)}_{n+h}), \cdots, g(\hat{Y}^{(M)}_{n+h})\}$.
Step 5  Repeat the above steps with different $\alpha$ values from $\{\alpha_k;\ k = 1, \cdots, K\}$ to get $K$ prediction results.
```

If we want to apply the GA-NoVaS-without-a_0 method, we just need to change the Algorithm 1 a little bit. The difference between Algorithm 1 and Algorithm 2 is the a_0 term being removed. The optimal combination of α, a_1, b_1 is still selected based on the normalizing and variance-stabilizing purpose.

Algorithm 2: the h-step ahead prediction for the GA-NoVaS-without-a_0 method

```
Step 1     Define a grid of possible $\alpha$ values, $\{\alpha_k;\ k = 1, \cdots, K\}$, two grids of possible $a_1$, $b_1$ values. Fix $\alpha = \alpha_k$, then calculate the optimal combination of $\alpha_k, a_1, b_1$ of the GA-NoVaS-without-$a_0$ method.
Steps 2-5  Same as Algorithm 1, but the $\{W^*_{n+1}, \cdots, W^*_{n+h}\}$ are plugged into the analytic form of Eq. (3.22) and the standard normal distribution does not need to be truncated.
```

Remark 3 (The issue of applying Algorithms 1 and 2 while dealing with real datasets)**.** *For predicting real-life econometric data, although we can get K predictions from either Algorithm 1 or Algorithm 2, we do not know which prediction result can best depict future value. In this situation, one thing may be useful is that using part of available data to do "training" (such as cross validation) for finding the optimal α value. However, since the econometric series may be time-varying, the optimal α value for past data may not be suitable for predicting future situation. Moreover, it is also hard to select sample data to do "training". If we sample data randomly, it will destroy the time structure within these data points. If we sample data continuously, the optimal α value may be effective only for this period data. Therefore,* *determining an optimal α value of the NoVaS method to do forecasting remains an open question and we keep the discussion for a separate venue.*

4 Simulation

In this section, we perform extensive simulation studies to

- explore whether NoVaS-type methods can sustain superior performance over the model-based ones in a long-term forecasting scenario and
- if our new proposals can outperform the existing GE-NoVaS for both short- and long-term forecasting

For the first 5 simulation settings, we use the GARCH(1,1) models with different parameters and error distributions to generate data. For the last three settings, we generate data from EGARCH(1,1) and GJR-GARCH(1,1) models to check whether our new methods can adapt to model misspecification. Finally, to control for dependence of prediction performances on the length of datasets, we present results for all these settings in two sizes of data. What we do for comparing prediction abilities of different methods is conducting POOS predictions using simulated data and comparing best results (best forecasting performance over a grid of free tuning parameter α) of each method. This means we have a computationally heavy approach to compare different methods' potentially best performance. However, it also means we wanted to challenge the newly proposed methods at a maximum level, so as to see if they can beat even the best-performing scenarios of the current state-of-art methods.

4.1 Simulation Settings

In the simulation, 16 datasets (2 from each settings) are generated from different GARCH(1,1)-type models separately and the size of each dataset is 250 (smaller data mimics one year of econometric data) or 500 (larger data mimics two years of econometric data).

Model 1: Time-varying GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = \omega_{0,t} + \beta_{1,t}\sigma_{t-1}^2 + \alpha_{1,t}X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
$g_t = t/n; \omega_{0,t} = -4sin(0.5\pi g_t) + 5; \alpha_{1,t} = -1(g_t - 0.3)^2 + 0.5; \beta_{1,t} = 0.2sin(0.5\pi g_t) + 0.2,\ n = 250$ or 500
Model 2: Another Time-varying GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + \beta_{1,t}\sigma_{t-1}^2 + \alpha_{1,t}X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
$\alpha_{1,t} = 0.1 - 0.05t/n;\ \beta_{1,t} = 0.73 + 0.2t/n,\ n = 250$ or 500
Model 3: Standard GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + 0.73\sigma_{t-1}^2 + 0.1X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
Model 4: Standard GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + 0.8895\sigma_{t-1}^2 + 0.1X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
Model 5: Standard GARCH(1,1) with Student-t errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + 0.73\sigma_{t-1}^2 + 0.1X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ t$ distribution with five degrees of freedom
Model 6: Exponential GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \log(\sigma_t^2) = 0.00001 + 0.8895\log(\sigma_{t-1}^2) + 0.1\epsilon_{t-1} + 0.3(|\epsilon_{t-1}| - E|\epsilon_{t-1}|),\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
Model 7: GJR-GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + 0.5\sigma_{t-1}^2 + 0.5X_{t-1}^2 - 0.5I_{t-1}X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
$I_t = 1$ if $X_t \leq 0; I_t = 0$ otherwise
Model 8: Another GJR-GARCH(1,1) with Gaussian errors
$X_t = \sigma_t\epsilon_t,\ \sigma_t^2 = 0.00001 + 0.73\sigma_{t-1}^2 + 0.1X_{t-1}^2 + 0.3I_{t-1}X_{t-1}^2,\ \{\epsilon_t\} \sim i.i.d.\ N(0,1)$
$I_t = 1$ if $X_t \leq 0; I_t = 0$ otherwise

The Model 1 presents the time-varying GARCH model by changing the a_0, a_1, b_1 slowly as time passing. Models 2, 3, 4 and 5 come from the simulation settings of (Chen and Politis, 2019) and present the GARCH(1,1) model with different error distributions and coefficients of σ^2 term. Models 6, 7 and 8 present different types of GARCH models.

Using these datasets, we perform 1-step, 5-steps and 30-steps ahead time aggregated POOS predictions as we did in Section 3.2. For measuring different methods' prediction performance on larger datasets (i.e., data size is 500), we use 250 data as a window to do predictions and roll this window through the whole dataset, then compare predictions with realized values based on Eq. (3.18). For evaluating different methods' performance on smaller datasets (i.e., data size is 250), we use 100 data as a window (more details of this procedure can be found on Section 3.2).

Note that we can perform the GE-NoVaS, GE-NoVaS-without-a_0, GA-NoVaS and GA-NoVaS without-a_0 methods to predict Y_{n+h} by generating $\{W^*_{n+1}, \cdots, W^*_{n+h}\}$ from a trimmed standard normal distribution or the empirical distribution of $\{W_t\}$, then we can calculate the optimal predictor based on L_1 or L_2 risk criterion. It means each NoVaS-type method possesses four variants based on different evaluation metrics and generation processes to perform predictions. In other words, there are total sixteen different NoVaS variants to do predictions. We also add one more Model-free method–GARCH-bootstrap[9]–to do comparisons. This GARCH-bootstrap method has two variants (under L_1 or L_2 risk criterion) to do predictions. Additionally, we set the benchmark method as fitting one GARCH model directly (GARCH-direct).

In summary, nineteen different approaches are performed to do predictions using 16 different simulated datasets. Since we perform POOS forecasting here and do not know which α is optimal, we perform every NoVaS variants using α from eight potential values $\{0.1, 0.2, \cdots, 0.8\}$ and then select the best result. For simplifying the presentation of results, we further select the best predictions from four best results of each NoVaS method (recall that each NoVaS method possesses four variants to perform predictions) and use this optimal result to be the best prediction to which each NoVaS method can reach.

4.2 Simulation results for larger sample size

In this subsection, we compare the performance of our new methods with GARCH-type methods and the existing GE-NoVaS method on forecasting with large datasets.

4.2.1 Simulation results of Model 1–5

First let us analyze prediction results of different methods using larger datasets generated from Model 1–5. From Table 1, we clearly find NoVaS-type methods outperform the GARCH-direct and GARCH-bootstrap methods. Especially for using the Model 1 data to do 30-steps ahead time aggregated predictions, the performance of GARCH-direct method is terrible. All Model-free methods are almost 30 times better than the GARCH-direct method. This means that the normal prediction method may be spoiled by error accumulation problem when long-term predictions are required.

In addition to the overall advantage of NoVaS-type methods over GARCH-type methods[10], we find two proposals proposed in this article are all useful since almost all best methods for predicting different datasets come from either the GE-NoVaS-without-a_0 method or GA-NoVaS-type (GA-NoVaS and GA-NoVaS-without-a_0) methods. For example, for 30-steps ahead time aggregated predictions of Models 3 and 5 data, the GA-NoVaS-without-a_0 and GA-NoVaS methods are around 20% and 50% better than the GE-NoVaS method, respectively. Additionally, the GE-NoVaS-without-a_0 method also outperforms the GE-NoVaS method for almost all cases.

Moreover, we can find the GE-NoVaS method is even worse than the GARCH-direct method for the case of 30-steps ahead predictions of Model 3 data. On the other hand, refined NoVaS methods via our proposals are more robust than the existing GE-NoVaS method.

[9] The main idea within this method is fitting a GARCH(1,1) model and using this model to express the analytical form of future data like we did in GE-NoVaS and GA-NoVaS methods. Then we bootstrap M times $\{\sigma^*_{n+1}, \cdots, \sigma^*_{n+h}\}$ from the empirical distribution of $\{\sigma_t\}$. Plug these bootstrapped σ^* into the predictor of future data to get M pseudo-values of predictions. Finally, taking mean or median of these pseudo-values, we can get optimal predictions based on L_1 or L_2 risk criterion, see Chen and Politis (2019) for more details.

[10] In this paper, GARCH-type methods represent the GARCH-bootstrap and GARCH-direct methods.

4.2.2 Different GARCH specifications

Since the main crux of a model-free method is how such nonparametric methods are robust to underlying data-generation process, here we explore other GARCH-type data generations. Note that two of our new methods are GARCH based. So it is interesting to explore whether even these methods can sustain a different type of true underlying generation and can in general improve upon existing methods.

We can get the answer from prediction results of using Model 6–8 in Table 1. The NoVaS-type methods still outperform GARCH-type methods for these cases. Although the GE-NoVaS method is the best method for two cases, the performance is only marginally better than new methods. On the other hand, for the cases of using data generated from Model 6, the GE-NoVaS-without-a_0 method significantly outperform the GE-NoVaS method. It creates around 10%, 19% and 20% improvements on 1-step, 5-steps and 30-steps ahead predictions, respectively. Additionally, the GA-NoVaS-without-a_0 method is also better than the GE-NoVaS method for almost all cases. Therefore, generally speaking, when forecast is performed with a large simulated dataset, the NoVaS-type methods can sustain different types of true underlying models. Moreover, new methods are superior than the existing GE-NoVaS method.

Table 1: Comparison results of using 500 simulated data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
M1-1step	0.89258	0.85483	0.88735	**0.84138**	0.87868	1.00000
M1-5steps	0.40603	0.40510	0.40296	**0.40137**	0.41999	1.00000
M1-30steps	0.03368	0.03293	0.03294	**0.03290**	0.03997	1.00000
M2-1step	**0.95689**	0.98431	0.96069	0.99658	1.02825	1.00000
M2-5steps	0.89981	0.90156	**0.89739**	0.91980	1.04884	1.00000
M2-30steps	0.63126	0.49761	0.64042	**0.48396**	1.09037	1.00000
M3-1step	0.99938	0.99360	1.00150	**0.98407**	1.00106	1.00000
M3-5steps	0.98206	0.94456	0.96088	**0.94073**	0.95810	1.00000
M3-30steps	1.10509	0.92128	1.03683	**0.90855**	0.95699	1.00000
M4-1step	0.98713	**0.97484**	0.98466	0.99640	1.03154	1.00000
M4-5steps	0.95382	**0.90862**	0.95362	0.95338	1.05703	1.00000
M4-30steps	0.75811	0.68266	0.69208	**0.67594**	1.01786	1.00000
M5-1step	0.96940	0.96936	**0.94066**	0.97151	1.01598	1.00000
M5-5steps	0.84751	0.82810	**0.72806**	0.82747	0.98493	1.00000
M5-30steps	0.49669	0.53026	**0.24318**	0.47311	0.78657	1.00000
M6-1step	1.00175	**0.90196**	1.00514	0.93509	0.92102	1.00000
M6-5steps	0.93796	0.75922	0.94249	0.80311	**0.73595**	1.00000
M6-30steps	0.50740	**0.40790**	0.51350	0.41112	0.54646	1.00000
M7-1step	0.98857	0.96081	0.98737	**0.95932**	0.97881	1.00000
M7-5steps	0.85539	**0.84889**	0.85371	0.85127	0.95994	1.00000
M7-30steps	**0.68202**	0.69698	0.68314	0.71391	0.87935	1.00000
M8-1step	0.96001	**0.92557**	0.96463	0.93452	0.98151	1.00000
M8-5steps	0.97019	**0.92549**	0.98184	0.93178	1.01225	1.00000
M8-30steps	**0.30593**	0.33834	0.31813	0.33853	0.42410	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method. “Mi-j”steps denotes using data generated from the Model i to do j steps ahead time aggregated predictions.

4.3 Simulation results for shorter sample size

In this subsection we challenge our new methods in contrast to the GARCH based methods and the existing GE-NoVaS for small datasets. Doing better prediction with past data that is shorter in size is always considered a significant challenge and thus it was important to see if our methods can be as good or even improve from the existing ones.

All simulation results are in Table 2. Similar with results in Section 4.2, the NoVaS-type methods are still notably better than GARCH-type methods for all shorter samples. In addition to this expected result, new methods beat the existing GE-NoVaS method for all cases. In other words, these new methods are not influenced by negative effects of short sample size. Inversely, it seems these new methods can even bring more significant improvement doing prediction with short datasets.

Additionally, one thing should be noticed is that the GE-NoVaS method is beaten by the benchmark method for 30-steps ahead predictions of data generated from Model 3. On the other hand, new methods are all better than the benchmark method.

Table 2: Comparison results of using 250 simulated data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
M1-1step	0.91538	0.83168	0.91120	**0.83034**	0.87348	1.00000
M1-5steps	0.49169	0.43772	0.48479	**0.43247**	0.46654	1.00000
M1-30steps	0.25009	**0.22659**	0.24752	0.23035	0.23254	1.00000
M2-1step	0.91369	0.88781	0.91574	**0.87614**	0.95533	1.00000
M2-5steps	0.61001	0.52872	0.61094	**0.51712**	0.72210	1.00000
M2-30steps	0.77250	**0.73604**	0.74083	0.75251	0.78393	1.00000
M3-1step	0.97796	0.94635	0.96632	**0.93693**	1.01978	1.00000
M3-5steps	0.98127	**0.96361**	0.97897	0.99977	1.02789	1.00000
M3-30steps	1.38353	0.98872	**0.89001**	0.99818	0.93789	1.00000
M4-1step	0.99183	0.92829	0.95698	**0.92811**	0.97230	1.00000
M4-5steps	0.77088	**0.67482**	0.72882	0.67894	0.79454	1.00000
M4-30steps	0.79672	0.71003	**0.60950**	0.81115	1.01542	1.00000
M5,1y-1step	0.83631	**0.78087**	0.84134	0.79075	0.83749	1.00000
M5-5steps	0.38296	**0.34396**	0.38034	0.35155	0.41618	1.00000
M5-30steps	0.00199	0.00201	0.00200	**0.00194**	0.00277	1.00000
M6-1step	0.95939	0.94661	0.96499	**0.93863**	0.99458	1.00000
M6-5steps	0.93594	**0.84719**	0.97101	0.85851	0.98952	1.00000
M6-30steps	0.84401	0.70301	**0.67272**	0.70420	0.82080	1.00000
M7-1step	0.84813	**0.73553**	0.83628	0.83216	0.86427	1.00000
M7-5steps	0.50849	**0.46618**	0.50126	0.48020	0.59051	1.00000
M7-30steps	0.06832	**0.06479**	0.06817	0.06507	0.07509	1.00000
M8-1step	0.79561	**0.76586**	0.79994	0.83340	0.85211	1.00000
M8-5steps	0.48028	**0.38107**	0.47244	0.45665	0.45836	1.00000
M8-30steps	0.00977	**0.00918**	0.00942	0.00983	0.01039	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method. "*Mi-j*"steps denotes using data generated from the Model *i* to do *j* steps ahead time aggregated predictions.

5 Real-world data analysis

Previous work has already substantiated that NoVaS-type methods possess great prediction performance for some situations, such as one-step ahead predictions of volatility of financial returns and 5-steps ahead predictions of simulated data (Chen and Politis, 2019; Gulay and Emec, 2018). Additionally, from Section 4, we have found that NoVaS-type methods have great performance on dealing with different simulated data even for long-term predictions. However, whether NoVaS-type methods can sustain great performance on long-term predictions of real-world data is needed to be checked. In addition to this task, we still want to verify whether our new methods still keep superior performance.

For performing an extensive analysis and subsequently acquiring a convincing conclusion, we use worldwide stock price, index and currency data to do predictions. Moreover, we still apply two different lengths of data in our analysis. For large datasets (2-year period data), we take new datasets which come from Jan.2018 to Dec.2019 and old datasets which come from around 20 years ago, separately. Subsequently, we challenge our methods using shorter data size (1-year period data). Moreover, for testing different methods' prediction performance on the volatile period, we also do comparisons using most recent 1 year data (i.e., data from Nov. 2019 to Oct. 2020 which presents high volatile phenomenon because of the pandemic).

5.1 2-year period for old and new data

We adopt 2-year period of stock price datasets AAPL, BAC, MSFT and MCD to do forecasting. Taking 500 continuing time points from each data sets and transforming these price data to log-returns based on Eq. (3.16), then we perform similar procedure as which we did in Sections 3.2 and 4.1 to calculate the performance of each method and do comparisons.

In summary, we still compare six different methods' performance on 1-step, 5-steps and 30-steps ahead POOS time aggregated forecasting. All results are shown in Table 3. We can clearly find NoVaS-type methods still outperform GARCH-type methods no matter using old or new datasets. Combining this finding with the conclusion in Section 4.2, for the prediction using enough large datasets, we can safely conclude that applying NoVaS-type methods can avoid error accumulation problem and these methods have better performance than GARCH-type methods.

Additionally, from Table 3, we find most of best results for different cases come from using the GE-NoVaS-without-a_0 or GA-NoVaS-without-a_0 method. For example, the GE-NoVaS-without-a_0 and GA-NoVaS-without-a_0 methods can create around 20% and 30% improvement compared to GE-NoVaS on 30-steps ahead predictions of old BAC and MSFT stock price data, respectively. Although the GE-NoVaS method is the best choice for two cases, the performance of the GE-NoVaS and new methods are almost indistinguishable within these cases. Moreover, we can notice that the GE-NoVaS method is significantly worse than the GARCH-direct method for the 30-steps ahead predictions of old MCD data. On the other hand, NoVaS-type methods without a_0 term are more robust than the existing GE-NoVaS method. The GA-NoVaS method is also more robust than the GE-NoVaS method. One thing should be noticed is that the advantage of our new methods over the existing GE-NoVaS method prevail in both old and new datasets.

5.2 Normal 1-year period

For challenging our new methods in contrast to other methods for small real-life datasets, we separate every new 2-year period data in Section 5.1 to two equal 1-year period datasets, i.e., separate four new stock price datasets to eight samples. Additionally, we add 7 index data: Nasdaq, NYSE, Small Cap, Dow Jones, S&P 500 , BSE and BIST; and two stock price data: Tesla and Bitcoin into our analysis.

From Table 4 which presents the prediction results of different methods on 2018 and 2019 stock price data, we still observe that NoVaS-type methods outperform GARCH-type methods for most cases. Additionally, among different NoVaS methods, it is clear that our new methods are superior than the existing GE-NoVaS method, which implies our two proposals are useful. For example, for the case of 30-steps ahead predictions of 2018 BAC data, the GE-NoVaS-without-a_0 method brings around 40% improvement compared to the existing GE-NoVaS method. Additionally, for 30-steps ahead predictions of 2018 BAC and Tesla data, 2019 BAC, MCD and Tesla data, the existing NoVaS method is even worse than the GARCH-direct method. On the other hand, removing a_0 makes NoVaS-type methods be the best method or close to the best method for these cases, i.e., making NoVaS-type methods more robust. In addition to these positive findings, we find that the GA-NoVaS method is also better than the GE-NoVaS method for most cases, and it is also more robust than the GE-NoVaS method on cases where GE-NoVaS method's performance is terrible, such as 30-steps ahead predictions of 2019 MCD data, 2018 BAC and 2019 Tesla data.

From Table 5 which presents the prediction results of different methods on 2018 and 2019 index data, we can get exactly same conclusion that NoVaS-type methods are more superior than GARCH-type methods and our new NoVaS methods are more powerful than the existing GE-NoVaS method. For example, compared to the existing GE-NoVaS method, the GA-NoVaS-without-a_0 method can create around 60% improvement from the GE-NoVaS method on 30-steps ahead predictions of 2019 Bist data; the GA-NoVaS method can bring around 30% improvement from the GE-NoVaS method on 30-steps ahead predictions of 2018 Dow jones data. Moreover, the GE-NoVaS method is still beaten by the GARCH-direct method on some cases, such as for all different prediction steps of 2019 Small Cap index data. On the other hand, new methods are more robust on performing these cases.

Combining results presented in Tables 3 to 5, although it is hard to find a method which possesses uniformly better performance than other methods, we can safely conclude our new methods have generally better performance than the existing GE-NoVaS and GARCH-type methods on dealing with small and large real-life data. Additionally, the improvement generated by new methods using shorter sample size (1-year data) is more notably than using larger sample size (2-year data).

Remark 4. *One interesting finding is that the GA-NoVaS method works better on 2018 data and the GE-NoVaS-without-a_0 method tends to generate better predictions for 2019 data. For the GA-NoVaS-without-a_0 method, it possesses more balanced performance for 2018 and 2019 data.*

5.3 Volatile last 1-year period

In this subsection, we perform POOS forecasting using most recent 1 year data (i.e., data from Nov. 2019 to Oct. 2020). We tactically choose this period of time to evaluate whether our methods can sustain the extreme volatility introduced by the pandemic Covid-19. Stock market data and indices data worldwide was hit poorly and such instability in the system can make traditional model based predictions fail. Since our main goal in this paper is to boost model-free predictions we wanted to challenge our methods to check whether it can self-adapt to such structural incoherence between pre- and post-pandemic. For observing the affects of pandemic, we can take the price of SP500 index as an example. From Fig. 3, it is clearly that the price grew slowly during normal period Jan.2017 to Dec.2017. However, during the most recent one year, the price fluctuated severely due to pandemic.

Some studies already warned that there would be a period of volatility growth. The volatility of global financial market is even comparable with financial crisis of 2008. (Abodunrin et al., 2020; Fernandes, 2020; Yueh, 2020) Not surprisingly, for modeling appropriate process of volatility during fluctuated period, researchers developed different approaches from primary model–GARCH–to achieve this goal, such as Kim et al. (2011) found time series models with stable and tempered stable innovations had better predictive power in measuring market risk than standard models based on the normal distribution assumption during highly volatile period; Ben Nasr et al. (2014) applied a fractionally integrated time-varying GARCH (FITVGARCH) model to fit volatility of returns of Dow Jones Islamic Market World Index and showed it had better performance than the FIGARCH model; Karmakar and Roy (2020) developed a Bayesian method to estimate time-varying analogues of ARCH, GARCH and iGARCH models for describing frequent volatility changes over time due to market variability.

In this subsection, we focus on evaluating the performance of Model-free methods on handling volatile data and comparing the performance of the existing GE-NoVaS method and new NoVaS methods. For executing a comprehensive analysis, we investigate different methods' performance on stock price, index and currency data.

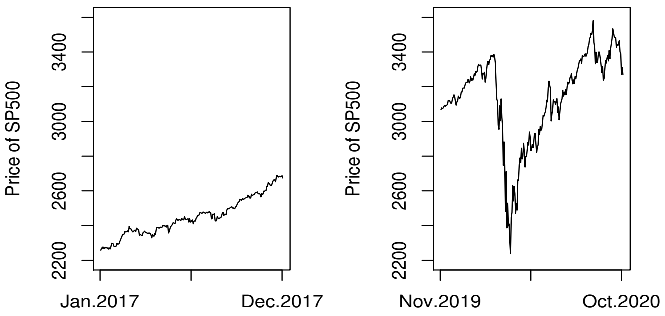

Figure 3: The left subfigure depicts the price of SP500 from Jan.2017 to Dec.2017 which presents a slow growth; The right subfigure depicts the price of SP500 from Nov.2019 to Oct.2020

5.3.1 Stock data

Nine most recent 1-year stock price datasets (usually 252 time points) are used to do predictions with different methods. The results are presented in Table 6.

The first thing we can notice is that NoVaS-type methods still outperform GARCH-type methods. This means NoVaS-type methods are more suitable to deal with volatile stock price data than GARCH-type methods to some extent. In addition to this overall advantage of NoVaS-type methods on GARCH-type methods, we can find best methods for almost all cases fall into three new methods. In other words, this phenomenon is a strong evidence to support the usefulness of our proposals.

Moreover, the improvement created by new methods are notable. Take 30-steps ahead predictions of MSFT stock price data as an example, we can find NoVaS-type methods are much better than GARCH-direct method. Among different NoVaS methods, the GE-NoVaS-without-a_0 method is around 45% better than the the GE-NoVaS method, and the GA-NoVaS-without-a_0 also creates a rough 40% improvement. In addition to this dataset, for 30-steps ahead predictions of NKE stock price data, the GA-NoVaS-without-a_0 and GE-NoVaS-without-a_0 methods generate around 30% and 20% improvements compared to the the GE-NoVaS method, respectively. For 30-steps ahead predictions of MCD stock price data, the GE-NoVaS-without-a_0 and GA-NoVaS-without-a_0 methods all bring around 77% improvements compared to the GE-NoVaS method.

5.3.2 Currency data

Six most recent 1-year currency datasets (usually 261 time points) are used to do predictions with different methods. The results are presented in Table 7.

We can find NoVaS-type methods still keep the superior place compared to GARCH-type methods on predicting currency data. Additionally, new proposed methods have generally better performance than the GE-NoVaS and GARCH-type methods. Although the improvement generated by new methods is slight for most cases, some cases can still render significant improvement, such as the GA-NoVaS and GE-NoVaS-without-a_0 methods create around 40% and 30% improvement compared to GE-NoVaS method on 30-steps ahead predictions of CNYJPY and CADJPY currency rate data, respectively.

One thing should be noticed is that Fryzlewicz et al. (2008) implied the ARCH framework seems to be a superior stationary GARCH methodology for dealing with the currency exchange data. Therefore, it is not surprising that our new methods can not introduce much significant improvement since the existing GE-NoVaS method is built based on the ARCH framework and the GA-NoVaS method is motivated by the GARCH model. On the other aspect, these slight improvement can also be seen as an evidence to show GA-NoVaS-type methods are robust to model misspecification.

5.3.3 Index data

Ten most recent 1-year index datasets (usually around 250–260 time points) are used to do predictions with different methods. The results are presented in Table 8.

Similar with conclusions corresponding to previous two classes of data, NoVaS-type methods still have obviously better performance than GARCH-type methods on predicting index data. In addition to this consistent advantage of NoVaS methods, new methods still dominate the existing GE-NoVaS method. For example, compared to the GE-NoVaS method, the GE-NoVaS-without-a_0 method creates around 70% improvement for Bovespa 30-steps and Dow Jones 30-steps cases. It also creates around 50% improvement on 30-steps ahead predictions of Nasdaq data; the GA-NoVaS-without-a_0 method creates around 50% improvement for NYSE 30-steps and SmallCap 30-steps cases, around 30% improvement for BSE 30-steps case.

In addition to these expected results, we find the GE-NoVaS method is even 14% worse than the GARCH-direct method for USDX future 1-step case. Although GA-NoVaS-type methods are also little bit worse than the benchmark method and GE-NoVaS-without-a_0 is only slightly better than the benchmark method, these methods are almost indistinguishable for this case. Therefore, excluding a_0 term from the existing GE-NoVaS transformation seems to be an approach which makes NoVaS methods more robust. This phenomenon is also consistent with our findings in Sections 4.2.1, 4.2.2, 4.3, 5.1 and 5.2.

5.4 Summary of real-world data analysis

After performing extensive real-world data analysis, we can conclude that NoVaS-type methods have generally better performance than GARCH-type methods. This advantage of NoVaS-type methods prevails in wide situations, such as doing predictions for either shorter or larger data, data of normal or volatile period. In addition to this encouraging results, three new NoVaS methods proposed in this paper all have generally better performance than the existing GE-NoVaS method. Although it is hard to determine which new method occupies the best performance for all different types of datasets, we can still make some suggestions for users to apply these various NoVaS methods based on all prediction results we have:

- For forecasting stock price using a large data (i.e., corresponding to Section 5.1), we recommend the GE-NoVaS-without-a_0 method to be the first choice.
- For forecasting stock price and index using a small data (i.e., corresponding to Section 5.2), we recommend the GA-NoVaS-without-a_0 method to be the first choice.
- For forecasting stock price during volatile period (i.e., corresponding to Section 5.3.1), we recommend the GA-NoVaS method to be the first choice.
- For forecasting currency during volatile period (i.e., corresponding to Section 5.3.2), we recommend the GA-NoVaS-without-a_0 method to be the first choice.
- For forecasting index during volatile period (i.e., corresponding to Section 5.3.3), we recommend the GA-NoVaS method to be the first choice.

Note that these recommendations are based on comparison results in this paper. In practice, a more reasonable way to apply NoVaS methods is selecting the optimal method among different NoVaS variants by some training procedure.

Table 3: Comparison results of using old and new 2-year period data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
Data of 20 years ago						
AAPL-1step	0.99795	0.98261	0.99236	**0.97836**	1.00482	1.00000
AAPL-5steps	1.04919	1.00090	1.04800	**0.96999**	1.02016	1.00000
AAPL-30steps	1.12563	0.97326	1.21986	**0.96174**	1.00881	1.00000
BAC-1step	0.99889	1.00416	1.00396	1.02780	**0.99648**	1.00000
BAC-5steps	1.04424	**0.97542**	1.02185	0.99399	0.99072	1.00000
BAC-30steps	1.32452	**0.99533**	1.13887	1.00363	1.05599	1.00000
MSFT-1step	0.98785	0.97376	0.98598	**0.96185**	0.98974	1.00000
MSFT-5steps	1.00236	0.96601	1.00096	**0.95271**	0.98403	1.00000
MSFT-30steps	1.25272	0.91046	1.09881	**0.88515**	1.12356	1.00000
MCD-1step	1.01845	**0.98873**	1.00789	0.99005	0.99456	1.00000
MCD-5steps	1.11249	0.98059	1.07748	**0.97777**	0.99387	1.00000
MCD-30steps	1.76385	0.99512	1.69757	0.99418	**0.96697**	1.00000
Data of 2018-2019						
AAPL-1step	0.80150	**0.78972**	0.79899	0.79915	0.84438	1.00000
AAPL-5steps	0.41405	**0.40260**	0.42338	0.40427	0.53249	1.00000
AAPL-30steps	**0.13207**	0.14771	0.14046	0.14543	0.30598	1.00000
BAC-1step	0.98393	0.97966	0.99164	**0.96542**	0.99408	1.00000
BAC-5steps	0.98885	0.95124	1.01480	**0.91857**	0.95791	1.00000
BAC-30steps	1.14111	**0.87414**	1.03657	0.88596	0.92458	1.00000
MSFT-1step	0.98405	0.97050	0.98630	**0.96374**	1.05370	1.00000
MSFT-5steps	0.65027	**0.64007**	0.67005	0.64278	0.84080	1.00000
MSFT-30steps	**0.19767**	0.20995	0.20060	0.21473	0.42449	1.00000
MCD-1step	0.99631	0.99614	0.99539	**0.98035**	1.00726	1.00000
MCD-5steps	0.95403	0.92120	0.95327	**0.91317**	1.00700	1.00000
MCD-30steps	0.75730	**0.62618**	0.75361	0.74557	0.95968	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method.

Table 4: Comparison results of using stock price data of 2018 and 2019

	GE-NoVaS	GE-NoVaS	GA-NoVaS	GA-NoVaS	GARCH	GARCH
		without a_0		without a_0	bootstrap	direct
MCD18-1step	0.98514	0.95332	0.97887	**0.94412**	1.02130	1.00000
MCD18-5steps	1.02720	0.88773	1.02519	**0.88151**	0.94535	1.00000
MCD18-30steps	0.62614	**0.61097**	0.63992	0.61153	0.89909	1.00000
MCD19-1step	0.95959	**0.93141**	0.96348	0.94559	0.97881	1.00000
MCD19-5steps	1.00723	**0.90061**	1.01169	0.90602	0.92116	1.00000
MCD19-30steps	1.05239	0.80805	0.95714	**0.77976**	0.80057	1.00000
AAPL18-1step	0.92014	0.96311	0.92317	**0.89283**	1.00288	1.00000
AAPL18-5steps	0.84798	0.85114	0.73461	**0.71233**	1.02531	1.00000
AAPL18-30steps	0.38612	**0.35731**	0.36324	0.37081	0.99618	1.00000
AAPL19-1step	0.84533	0.80948	0.81326	0.81872	**0.80374**	1.00000
AAPL19-5steps	0.85401	0.68191	0.79254	0.68792	**0.65603**	1.00000
AAPL19-30steps	0.99043	0.73823	0.99286	0.72892	**0.69184**	1.00000
BAC18-1step	0.94952	0.93031	0.93842	**0.92619**	1.04667	1.00000
BAC18-5steps	0.83395	0.86387	0.79158	**0.72512**	1.04852	1.00000
BAC18-30steps	1.34367	**0.81429**	0.90675	0.87630	1.23970	1.00000
BOA19-1step	1.04272	**0.97757**	1.04722	0.98605	1.00853	1.00000
BAC19-5steps	1.22761	**0.89571**	1.20195	0.95436	1.03207	1.00000
BAC19-30steps	1.45020	1.01175	1.41788	1.03482	**0.98703**	1.00000
MSFT18-1step	0.91705	0.94507	**0.90936**	0.95921	0.96877	1.00000
MSFT18-5steps	0.74553	0.76646	0.74267	**0.74237**	1.12363	1.00000
MSFT18-30steps	0.66990	0.68741	0.64770	**0.64717**	1.18625	1.00000
MSFT19-1step	1.03308	0.98469	1.00101	**0.95347**	1.00160	1.00000
MSFT19-5steps	1.22340	1.02387	1.18205	**0.95417**	1.00367	1.00000
MSFT19-30steps	1.23020	**0.97585**	1.21337	0.98476	1.14559	1.00000
Tesla18-1step	1.00181	0.95885	0.96074	**0.86238**	1.00783	1.00000
Tesla18-5steps	1.20383	1.02036	1.13335	1.01560	1.02205	**1.00000**
Tesla18-30steps	1.97328	1.22256	1.84871	1.25005	1.24363	**1.00000**
Tesla19-1step	1.00428	**0.98646**	1.01934	0.98955	0.99649	1.00000
Tesla19-5steps	1.06610	0.97523	1.07506	0.96107	**0.95072**	1.00000
Tesla19-30steps	2.00623	0.87158	1.71782	0.84366	**0.79804**	1.00000
Bitcoin18-1step	0.99636	0.99967	1.01731	**0.97734**	1.03637	1.00000
Bitcoin18-5steps	1.02021	1.02008	1.11880	**0.93826**	1.28683	1.00000
Bitcoin18-30steps	**0.86649**	0.95020	0.95506	0.91364	1.22959	1.00000
Bitcoin19-1step	0.89929	**0.86795**	0.88914	0.87256	1.00553	1.00000
Bitcoin19-5steps	0.62312	**0.55620**	0.63075	0.56789	0.80858	1.00000
Bitcoin19-30steps	0.00733	**0.00624**	0.00749	0.00631	0.01308	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method.

Table 5: Comparison results of using index data of 2018 and 2019

	GE-NoVaS	GE-NoVaS	GA-NoVaS	GA-NoVaS	GARCH	GARCH
		without a_0		without a_0	bootstrap	direct
Nasdaq18-1step	**0.91309**	0.94837	0.92303	0.92421	1.02178	1.00000
Nasdaq18-5steps	**0.76419**	0.83585	0.79718	0.78823	1.07972	1.00000
Nasdaq18-30steps	0.66520	0.74660	**0.65489**	0.67389	1.08343	1.00000
Nasdaq19-1step	0.99960	**0.93558**	0.98950	0.93843	0.99749	1.00000
Nasdaq19-5steps	1.15282	0.84459	1.09176	**0.84051**	1.01677	1.00000
Nasdaq19-30steps	0.68994	**0.58924**	0.69846	0.59218	0.86267	1.00000
NYSE18-1step	0.93509	0.95793	**0.93401**	0.96619	1.17770	1.00000
NYSE18-5steps	0.83725	0.87919	0.79330	**0.75822**	1.04958	1.00000
NYSE18-30steps	0.75053	0.79466	**0.61443**	0.61830	1.06842	1.00000
NYSE19-1step	0.92486	**0.90407**	0.91118	0.92193	0.97220	1.00000
NYSE19-5steps	0.86249	**0.69822**	0.82114	0.71038	0.90550	1.00000
NYSE19-30steps	0.22122	0.18173	0.22173	**0.18116**	0.24958	1.00000
Smallcap18-1step	**0.90546**	0.91299	0.91346	0.91101	1.00777	1.00000
Smallcap18-5steps	**0.72627**	0.73541	0.73955	0.73223	0.92849	1.00000
Samllcap18-30steps	0.50005	0.48461	0.46482	**0.46312**	1.12497	1.00000
Smallcap19-1step	1.02041	0.98731	1.00626	**0.98482**	1.02531	1.00000
Smallcap19-5steps	1.15868	0.87700	1.08929	**0.85490**	1.15878	1.00000
Samllcap19-30steps	1.30467	**0.88825**	1.28949	0.90360	0.93839	1.00000
Djones18-1step	0.90932	**0.84931**	0.90707	0.91192	0.98480	1.00000
Djones18-5steps	0.82480	0.86017	0.79965	**0.76226**	1.01144	1.00000
Djones18-30steps	0.72547	0.66418	**0.53021**	0.56854	1.01315	1.00000
Djones19-1step	0.96752	**0.96365**	0.96433	0.96977	1.05173	1.00000
Djones19-5steps	0.98725	**0.89542**	0.93315	0.91238	1.20414	1.00000
Djones19-30steps	0.86333	**0.80304**	0.85006	0.81803	1.15530	1.00000
SP50018-1step	0.91860	0.90469	0.91256	**0.88405**	1.10603	1.00000
SP50018-5steps	0.85108	0.90544	0.77305	**0.75646**	1.06950	1.00000
SP50018-30steps	0.88917	0.83210	**0.68156**	0.72104	1.11057	1.00000
SP50019-1step	0.96978	**0.92183**	0.96526	0.93162	1.01250	1.00000
SP50019-5steps	0.96704	**0.75579**	0.94028	0.77434	1.02589	1.00000
SP50019-30steps	0.34389	**0.29796**	0.34537	0.30127	0.41242	1.00000
BSE18-1step	0.99942	0.94676	**0.88322**	0.92568	0.98124	1.00000
BSE18-5steps	0.92061	0.82886	**0.78484**	0.84408	1.01296	1.00000
BSE18-30steps	0.52431	0.44818	**0.41010**	0.44092	0.49072	1.00000
BSE19-1step	0.70667	0.67694	0.70194	**0.66667**	0.67457	1.00000
BSE19-5steps	0.25675	0.23665	0.25897	**0.23603**	0.23874	1.00000
BSE19-30steps	0.03764	0.02890	0.03951	**0.02888**	0.02934	1.00000
Bist18-1step	0.93221	0.92271	**0.92215**	0.94138	0.96797	1.00000
Bist18-5steps	0.82149	0.80837	**0.79664**	0.81417	0.86740	1.00000
BIST18-30steps	1.34581	**1.09665**	1.42233	1.09900	1.18145	1.00000
BIST19-1step	0.96807	**0.95467**	0.97209	0.98234	1.02190	1.00000
BIST19-5steps	0.98944	**0.82898**	1.03903	0.85370	0.94036	1.00000
BIST19-30steps	2.21996	0.88511	2.10562	**0.85743**	0.91893	1.00000

Table 6: Comparison results of using most recent 1-year stock price data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
NKE-1step	0.63568	0.65295	**0.63209**	0.65594	0.70808	1.00000
NKE-5steps	0.20171	0.22550	**0.19089**	0.22226	0.27463	1.00000
NKE-30steps	0.00411	0.00337	**0.00278**	0.00340	0.00858	1.00000
AMZN-1step	0.97099	**0.90200**	0.96719	0.90487	1.00360	1.00000
AMZN-5steps	0.88705	**0.71789**	0.88274	0.72850	0.99648	1.00000
AMZN-30steps	0.58124	0.53460	0.62863	**0.53310**	0.99012	1.00000
IBM-1step	0.80222	0.80744	0.79823	**0.79509**	0.86276	1.00000
IBM-5steps	0.38933	0.40743	**0.37346**	0.38413	0.48312	1.00000
IBM-30steps	0.01143	0.00918	0.00996	**0.00879**	0.01319	1.00000
MSFT-1step	0.80133	0.84502	**0.79528**	0.81582	0.92811	1.00000
MSFT-5steps	0.35567	0.37528	**0.33419**	0.38022	0.56300	1.00000
MSFT-30steps	0.01342	**0.00732**	0.01031	0.00784	0.01133	1.00000
SBUX-1step	0.68206	0.69943	0.67067	**0.66743**	0.81002	1.00000
SBUX-5steps	0.24255	0.30528	**0.23072**	0.26856	0.40975	1.00000
SBUX-30steps	0.00499	0.00289	0.00337	**0.00236**	0.00525	1.00000
KO-1step	0.77906	0.81309	**0.75389**	0.77035	1.01416	1.00000
KO-5steps	0.34941	0.39679	**0.32459**	0.33405	0.83356	1.00000
KO-30steps	0.01820	0.01963	0.01848	**0.01582**	0.04650	1.00000
MCD-1step	0.51755	0.58018	**0.51351**	0.56414	0.64685	1.00000
MCD-5steps	0.10725	0.17887	**0.09714**	0.17439	0.28024	1.00000
MCD-30steps	3.32E-05	**7.48E-06**	2.97E-05	7.62E-06	9.60E-06	1.00E+00
Tesla-1step	0.90712	0.89253	0.90250	**0.88782**	0.96405	1.00000
Tesla-5steps	0.68450	**0.66177**	0.67935	0.66937	0.81825	1.00000
Tesla-30steps	**0.21643**	0.22460	0.21718	0.22395	0.30598	1.00000
Bitcoin-1step	0.36323	0.36346	**0.36260**	0.36326	0.36717	1.00000
Bitcoin-5steps	**0.01319**	0.01321	0.01321	0.01322	0.01356	1.00000
Bitcoin-30steps	7.75E-17	7.76E-17	**7.65E-17**	7.75E-17	8.08E-17	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method.

Table 7: Comparison results of using most recent 1-year currency data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
CADJPY-1step	0.46940	0.48712	**0.46382**	0.48367	0.53972	1.00000
CADJPY-5steps	0.11678	0.13549	**0.11620**	0.14376	0.18647	1.00000
CADJPY-30steps	0.00584	**0.00394**	0.00430	0.00482	0.01764	1.00000
EURJPY-1step	0.95093	**0.94206**	0.94682	0.95133	1.01152	1.00000
EURJPY-5steps	0.76182	0.76727	0.77091	**0.75636**	0.92545	1.00000
EURJPY-30steps	0.16202	**0.15350**	0.17956	0.18189	0.40889	1.00000
USDCNY-1step	0.98905	0.96766	0.97861	**0.95757**	0.98800	1.00000
USDCNY-5steps	0.93182	0.86364	0.92614	**0.83523**	0.90909	1.00000
USDCNY-30steps	0.57171	0.60121	**0.57100**	0.60131	0.68317	1.00000
GBPJPY-1step	0.86971	0.86553	**0.86474**	0.87160	0.89818	1.00000
GBPJPY-5steps	0.49749	0.49208	0.49612	**0.48842**	0.57267	1.00000
GBPJPY-30steps	0.17058	0.17246	**0.16987**	0.17262	0.28965	1.00000
USDINR-1step	0.97289	0.98975	0.96829	**0.93140**	0.99573	1.00000
USDINR-5steps	0.80866	0.81103	0.78008	**0.75693**	0.89675	1.00000
USDINR-30steps	**0.09725**	0.17966	0.09889	0.11380	0.22883	1.00000
CNYJPY-1step	0.77812	0.79877	0.77983	**0.74586**	0.85845	1.00000
CNYJPY-5steps	0.38875	0.40569	0.38407	**0.34839**	0.64200	1.00000
CNYJPY-30steps	0.08398	0.06270	**0.05240**	0.05444	0.07080	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method.

Table 8: Comparison results of using most recent 1-year index data

	GE-NoVaS	GE-NoVaS without a_0	GA-NoVaS	GA-NoVaS without a_0	GARCH bootstrap	GARCH direct
SP500-1step	0.97294	**0.92349**	0.95881	0.92854	1.00658	1.00000
SP500-5steps	0.96590	**0.75183**	0.94457	0.77060	1.03608	1.00000
SP500-30steps	0.34357	**0.29793**	0.34561	0.30115	0.41265	1.00000
Nasdaq-1step	0.71380	0.75350	**0.70589**	0.77753	0.84977	1.00000
Nasdaq-5steps	0.29332	0.33519	**0.27007**	0.36428	0.56791	1.00000
Nasdaq-30steps	0.01223	**0.00599**	0.00618	0.00696	0.00970	1.00000
NYSE-1step	0.55741	0.57174	0.55548	**0.54598**	0.71931	1.00000
NYSE-5steps	0.08994	0.10182	**0.07666**	0.07798	0.22302	1.00000
NYSE-30steps	1.36E-05	6.64E-06	9.06E-06	**6.57E-06**	3.06E-05	1.00000
SmallCap-1step	0.58170	0.60931	**0.57392**	0.57773	0.71171	1.00000
SmallCap-5steps	0.10270	0.10337	0.10135	**0.09628**	0.19302	1.00000
SmallCap-30steps	7.00E-05	5.96E-05	4.33E-05	**3.65E-05**	1.70E-04	1.00000
BSE-1step	0.39493	0.39745	**0.37991**	0.39851	0.40593	1.00000
BSE-5steps	0.03320	0.04109	**0.02829**	0.04170	0.05398	1.00000
BSE-30steps	2.45E-05	1.82E-05	2.19E-05	**1.73E-05**	6.48E-05	1.00000
DAX-1step	0.65372	**0.64727**	0.65663	0.66097	0.75091	1.00000
DAX-5steps	0.10997	**0.10356**	0.10828	0.11085	0.22833	1.00000
DAX-30steps	4.97E-05	7.37E-05	**4.87E-05**	7.81E-05	1.77E-04	1.00000
USDX future-1step	1.14621	**0.99640**	1.00926	1.03693	1.01271	1.00000
USDX future-5steps	0.61075	0.54834	0.53834	**0.51997**	0.75153	1.00000
USDX future-30steps	0.10723	0.10278	**0.09911**	0.10063	0.34850	1.00000
Bovespa-1step	0.60031	0.57558	**0.57316**	0.60656	0.69165	1.00000
Bovespa-5steps	0.08603	0.07447	**0.06201**	0.09395	0.09719	1.00000
Bovespa-30steps	6.87E-06	**2.04E-06**	2.82E-06	3.19E-06	4.27E-06	1.00000
Djones-1step	0.56357	0.57550	0.55020	**0.54422**	0.69082	1.00000
Djones-5steps	0.09810	0.11554	**0.08239**	0.08698	0.18591	1.00000
Djones-30steps	4.32E-05	**1.24E-05**	2.22E-05	2.65E-05	5.10E-05	1.00000
BIST-1step	0.94794	**0.90832**	0.95313	0.92418	0.99634	1.00000
BIST-5steps	0.48460	**0.47480**	0.49098	0.49279	0.71316	1.00000
BIST-30steps	**0.05478**	0.05550	0.05980	0.05671	0.16350	1.00000

Note: The benchmark is the GARCH-direct method. The performance values of each method are calculated from the best prediction result among different variants within each method.

6 Conclusion

In this paper, we show NoVaS methods can avoid error accumulation problem when long step ahead predictions are required. These methods outperform GARCH-type methods (GARCH-bootstrap and GARCH-direct methods) on predicting either simulated data or real-world data under different forecasting horizon. Moreover, NoVaS-type methods can sustain shorter sample size and the extreme volatility. Additionally, our two new proposals, which are building a novel GA-NoVaS transformation and removing a_0 term in the transformation structure, enhance the performance of NoVaS methods further. Based on our results presented in Sections 4 and 5, generally speaking, NoVaS methods without a_0 have better performance than the existing GE-NoVaS method. More specifically, we make some recommendations to users for helping them apply various NoVaS-type methods under different situations.

In the future, we plan to explore NoVaS method in different directions. Even if model-free methods have been showing superior performance, it is difficult to pin down on a single transformation that can single-handedly beat every other methods. Our new methods corroborate that and also open up avenues where one can explore other specific transformation structures to improve on the existing ones. There is also a lot of scope in proving statistical validity of such predictions. From a statistical inference point of view, one can construct prediction intervals for these predictions using bootstrap. Such prediction intervals are well sought in the econometrics literature and some results on asymptotic validity of these can be proved. Additionally, we can also explore dividing the datasets into test and training in some optimal way and see if that can improve performance of these methods. In the financial market, the stock data move together. So it would be exciting to see if one can do Model-free predictions in a multiple time series scenario.

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